Team Round Solutions

LMT Spring 2024

May 4, 2024

1. **[16]** How many positive divisors of 160 are not divisors of 36?

   Proposed by Benjamin Yin

   Solution. $\boxed{9}$

   We have $160 = 2^5 \cdot 5$ and $36 = 2^2 \cdot 3^2$. Divisors of 160 that are multiples of 5 will work (6 such numbers) and otherwise we have 2^3, 2^4, 2^5 only. This gives us an answer of $\boxed{9}$. ☐

2. **[18]** On May 4th, Yoda wakes up at a random time between 7 AM and 8 AM. On that same day, his planet will get invaded at a random time between 7:35 AM and 7:45 AM. What is the probability that Yoda wakes up before the invasion?

   Proposed by Aidan Duncan

   Solution. $\boxed{\dfrac{2}{3}}$

   If Yoda wakes up before 7:35 then he will always wake up before the invasion. This happens with probability $\frac{7}{12}$. If Yoda wakes up from 7:35 to 7:45 (probability $\frac{1}{6}$, there is a $\frac{1}{2}$ chance (by symmetry) that he wakes up before the invasion. In total the answer is $\frac{7}{12} + \frac{1}{12} = \boxed{\dfrac{2}{3}}$ ☐

3. **[20]** Find the sum of the solutions to the equation $(x+3)^3 + (x+2)^2 + (x+1)^1 + (x+0)^0 = 0$.

   Proposed by Edwin Zhao

   Solution. $\boxed{-10}$

   By Vieta we just need the coefficient of x^2. We see that it is 10 and the x^3 coefficient is 1, so the answer is $-\frac{10}{1} = \boxed{-10}$. ☐

4. **[22]** In quadrilateral $ABCD$, let $\angle ABD = 32^\circ$, $\angle CBD = 64^\circ$, $\angle CDB = 58^\circ$, and $\angle ADB = 74^\circ$. Find $\angle ACD$.

   Proposed by Derek Zhao

   Solution. $\boxed{16}$

   Notice $\angle BAD = 74^\circ$ and $\angle BCD = 58^\circ$. Therefore, $AB = BD = BC$, so $\angle ACD = 48^\circ$ and our answer is $\angle ACD = \boxed{16^\circ}$. ☐

5. **[24]** Andy starts with the number 2^{2024}. He can perform each of the following operations as many times as needed in any order. However an operation can only be used if the result would be an integer.

   The operations are:

   - Take the square root of the number.
   - Multiply the number by $\frac{5}{2}$.
   - Take the base-10 logarithm of the number.

Let m be the minimum achievable output after a series of operations. Find the minimum number of operations it takes Andy to produce m.

Proposed by Jacob Xu

Solution. $\boxed{256}$

We want our final operation to be a $\log_{10}$ operation to make the final value very small. First, take the square root twice first. Now our value is 2^{506}. Now we multiply this number by $\frac{5}{2}$ 253 times so that we get the number 10^{253}. Finally we take $\log_{10}$ of this number and we end up with 253. In total this takes $2+253+1 = \boxed{256}$ operations. □

6. **[26]** Peter begins at the origin on the xy-plane with a device that allows him to move 1 unit in a straight line in any direction. After using the device 3 times, Peter ends up at $(1,0)$. Let S denote the set of all points Peter could have passed through along his trip. Find the area of S.

Proposed by Peter Bai

Solution. $\boxed{2+\pi}$

First, notice that regardless of the direction of the first movement, it is always possible to get to $(1,0)$ in two more movements. As a result, S must contain the unit disk centered at the origin. By symmetry (consider the last movement), S must also contain the unit disk centered at $(1,0)$.

Next, notice that both the beginning and end points of the 2nd movement cannot be greater than one unit away from the x-axis. As a result, S cannot contain points above or below $y = \pm 1$. However, constructions such as the path $(0,0),(0,1),(1,1),(1,0)$ do reach all the way to these lines. Additionally, rotating the first movement clockwise from this construction results in the first two movements sweeping out the entirety of the unit square with vertices at $(0,0),(0,1),(1,1),(1,0)$. By symmetry, the reflection of this square over the x-axis is in S as well.

As a result, S is the union of two unit semicircles and a rectangle with side lengths of 1 and 2. Our answer is thus $1\cdot 2+2\cdot \frac{\pi}{2} = \boxed{2+\pi}$. □

7. **[28]** Find the number of ways to fill in the cells of a 3×3 grid with $\{1,1,2,2,3,3,4,4,5\}$ such that there are exactly 5 cells with strictly greater values than their neighbors (cells that share an edge).

Proposed by Muztaba Syed

Solution. $\boxed{204}$

First note that these 5 cells must be the corners and the center of the grid. If the 4 edge cells contain 1, 1, 2, 2 this will work, and there are $\binom{4}{2}\cdot\binom{5}{1}\cdot\binom{4}{2} = 180$ ways to assign the numbers here.

If the 4 edge cells not these values we see they must be 1, 1, 2, 3 by inspection. In this scenario there must be a 2 in a corner with 2 1s adjacent to it (4 options). There are then 2 options to assign the location for other 2, and this restricts the placement of the 3s. There are then 3 ways to arrange the 4s and 5, for a total of $4\cdot 2\cdot 3 = 24$. So in total our answer is $180+24 = \boxed{204}$. □

8. **[30]** Let $f(n)$ denote the least positive integer k such that k has exactly n positive integer divisors. Let

$$N = \sum_{n=1}^{100} f(n).$$

Find $\lfloor \log_2(N)\rfloor$.

Proposed by Muztaba Syed

Solution. $\boxed{96}$

If n is a prime $f(n) = 2^{n-1}$, so $f(97) = 2^{96}$. The next smallest prime is 89 and $f(89) = 2^{88}$ which is much smaller. When $n \le 89$ we see that $f(n)$ is smaller than 2^{88}. For composite values of $n > 89$ we see that $f(n)$ is quite small compared to 2^{n-1}. This is because we can write them as $2^a\cdot 3^b$ where a and b are small enough that this value is negligible compared to 2^{96}.

So for all values $n \ne 97$, $f(n) \le 2^{88}$ roughly. This means that the sum of these f values is at most $99\cdot 2^{88}$ which is smaller than 2^{96}. So our sum will be somewhere between 2^{96} and 2^{97}, so the answer is $\boxed{96}$. □

9. **[32]** Evin and his friends often think about *The Game*. Starting at noon, Evin thinks about The Game every 3 minutes on the dot, Jerry thinks about the game every 4 minutes on the dot, and Billiam thinks about the game every 6 minutes on the dot. Whenever any one of them thinks about The Game, there is a 50-50 chance that they announce "I lost the game", which causes each other friend to think about the game as well. What is the expected number of minutes from noon until the next time that all three friends think about the game simultaneously? (This could happen either because a friend announced that they lost the game, or because all three friends privately think about the game at the same instant.)

   Proposed by Evin Liang

   Solution. $\boxed{\dfrac{269}{64}}$

   Let T be the number of minutes until the next time they think about the game. By linearity of expectation,

$$E(T) = \sum_{i=1}^{\infty} \Pr(T \geq i).$$

   If $i > 12$, $\Pr(T \geq i) = 0$. So the answer is

$$1+1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}+\frac{1}{64}=\boxed{\frac{269}{64}}.$$

   □

10. **[34]** There are 2021 heads-up coins arranged in a circle, and Jelfin starts next to an arbitrary (heads-up) coin. For all positive integers i, after the ith second, Jelfin moves i coins in the clockwise direction and then flips the coin he's moved next to. After n seconds, Jelfin lands on a coin which is tails-up. Find the least possible value of n.

    Proposed by Muztaba Syed

    Solution. $\boxed{66}$

    If we start at coin 0 then after a seconds we are at coin $\frac{a(a+1)}{2} \pmod{2021}$. So if after a seconds and b seconds we are on the same coin then

$$\frac{a(a+1)}{2} \equiv \frac{b(b+1)}{2} \pmod{2021} \Rightarrow 2021 \mid (a-b)(a+b+1)$$

    The two values $a-b$ and $a+b+1$ have opposite parity, and multiply to a multiple of $43\cdot 47$. We need to minimize their sum, so we try 43 with 94 and 47 with 86. We see the latter case gives us a smaller value of $\boxed{66}$. □

11. **[38]** In triangle ABC, let $AB = 3$, $BC = 4$, and $CA = 5$. Define the Sakupen Circle $(X, \triangle XYZ)$ of a vertex X with respect to $\triangle XYZ$, as the circle passing through X, the midpoint of XY, and the midpoint of XZ. Let M, N, and P be the midpoints of sides AB, BC, and CA, respectively. Let $J \neq A$ be the intersection of segment AN and (A, ABC), and let $H \neq N$ be the intersection of segment AN and $(B, \triangle ABC)$. Find the area of quadrilateral $JMHP$.

    Proposed by Benjamin Yin

    Solution. $\boxed{\dfrac{12}{13}}$

    Claim: $JMHQ$ is a parallelogram.

    Proof: First we can prove $MJ \parallel PH$. We claim they are both perpendicular to JH. Indeed $\angle MJH = \angle MJN = 180^\circ - \angle MBN = 90^\circ$ and $\angle JHP = \angle AHP = \angle AMP = 90^\circ$.

    Next we can prove $JP \parallel MH$. We have

$$\angle JHM = \angle MPA = \angle MNB = \angle PBN = \angle BJN.$$

    The above angle equalities come from $AMHP$ cyclic, $\triangle AMP \cong \triangle MBN$, $MBNP$ is a rectangle, and $MJPN$ is cyclic. ■

    Let the intersection of segments HJ and PM be O. Then, we need to find MP, HJ, and $\angle HOM$ to finish the problem, because the area of a parallelogram is given by $\frac{1}{2}d_1 d_2 \sin(x)$, where d_1, d_2 are the lengths of the diagonals and x is the angle between them. $MP = 2$ because it is the midline parallel to side BC. We want to find the length of HJ, but since $MHPJ$ is a parallelogram, $HJ = 2HO$. Also, $MP = 2MO$.

Triangle AMO has side lengths $AM = \frac{3}{2}$ and $MO = 1$, with a right angle at M. H is also the foot of the perpendicular from M to side AO. So, we can find the length of MH, and then use Pythagorean Theorem to find HO. The area of

$$AMO = \frac{1}{2}AM \cdot MO = \frac{1}{2}\left(\frac{3}{2}\right)(1) = \frac{3}{4} = \frac{1}{2}AO \cdot MH.$$

Then we have

$$AO = \sqrt{1+\frac{9}{4}} = \frac{\sqrt{13}}{2},$$

so $\frac{3}{4} = \frac{1}{2}\left(\frac{\sqrt{13}}{2}\right)MH$,. This means $MH = \frac{3}{\sqrt{13}}$. By the Pythagorean Theorem, $MH^2+OH^2 = MO^2$, meaning $\frac{9}{13}+OH^2 = 1$, or $OH = \frac{2}{\sqrt{13}}$.

Finally, we need to find $\sin\angle HOM$. We have $\angle HOM = \angle MOA = \angle BNA$, so $\sin\angle HOM = \sin\angle BNA = \frac{AB}{AN} = \frac{3}{\sqrt{13}}$. So, the area of the parallelogram is

$$\frac{1}{2}\cdot 2\cdot\frac{2}{\sqrt{13}}\cdot 2\cdot\frac{3}{\sqrt{13}} = \boxed{\frac{12}{13}}.$$

□

12. **[42]** Let x, y, and z be real numbers satisfying

$$x^2+y^2 = 1+2yz-xy$$
$$z^2 = 2xz-xy-1$$

Find the minimum possible positive value of x.

Proposed by Evin Liang

Solution. $\boxed{\frac{2\sqrt{5}}{5}}$

Note that adding the two equations gives $(x+y-z)^2 = 0$, so $x+y-z = 0$. Then $z = x+y$, so the equations reduce to $y^2+xy+1-x^2 = 0$. The discriminant in y is $5x^2-4$, so $x \geq \frac{2\sqrt{5}}{5}$. Since $(x,y,z) = \left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ is a solution, the minimum positive value of x is $\boxed{\frac{2\sqrt{5}}{5}}$. □

13. **[46]** Let $\tau(n)$ denote the number of positive divisors of n. Let $\tau_1(n) = \tau(n)$ and for $k \geq 2$ let

$$\tau_k(n) = \sum_{d|n}\tau_{k-1}(d)\tau\left(\frac{n}{d}\right).$$

Find $\tau_5(2024)$.

Proposed by Muztaba Syed

Solution. $\boxed{22000}$

The main idea is that $\tau_k(n)$ is equal to the number of tuples $(a_1, a_2, \dots, a_{2k})$ such that $\prod_{i=1}^{2k} a_i = n$.

We can prove this with induction on k. The base case of $k = 1$, is clear (the number of divisors is the number of pairs (a,b) such that $ab = n$). For the inductive step assume this is true up to k. In a tuple of $2k+2$ elements with a product of n, the product of the first $2k$ elements can be any divisor d of n. The next two elements will have a product of $\frac{n}{d}$. The number of ways to do this is just $\tau_k(d)\tau\left(\frac{n}{d}\right)$, so summing this over all d gives the number of tuples. But this is just $\tau_{k+1}(n)$.

To finish note that $2024 = 2^3\cdot 11\cdot 23$, so by Stars and Bars the answer is

$$\binom{2\cdot5-1+3}{3}\binom{2\cdot5-1+1}{1}\binom{2\cdot5-1+1}{1} = \boxed{22000}.$$

□

14. **[50]** Evaluate

$$\tan\left(\sum_{x=1}^{100}\arctan\left(\frac{2}{x^2}\right)\right).$$

Proposed by Jerry Xu

Solution. $\boxed{-\frac{5150}{4949}}$

Given such a large summation, we can guess that the terms will telescope. Note that

$$\begin{aligned}\tan\left(\tan^{-1}(x+2)-\tan^{-1}(x)\right) &= \frac{\tan\left(\tan^{-1}(x+2)\right)-\tan\left(\tan^{-1}(x)\right)}{1+\tan\left(\tan^{-1}(x)\right)\tan\left(\tan^{-1}(x+2)\right)}\\ &= \frac{x+2-x}{1+x(x+2)}\\ &= \frac{2}{(x+1)^2}\end{aligned}$$

By tangent subtraction formula. Thus,

$$\begin{aligned}\tan^{-1}\left(\frac{2}{x^2}\right) &= \tan^{-1}\left(\tan\left(\tan^{-1}(x+1)-\tan^{-1}(x-1)\right)\right)\\ &= \tan^{-1}(x+1)-\tan^{-1}(x-1).\end{aligned}$$

Finally, we have that

$$\begin{aligned}\sum_{x=1}^{100}\arctan\left(\frac{2}{x^2}\right) &= \sum_{x=1}^{100}(\tan^{-1}(x+1)-\tan^{-1}(x-1))\\ &= \tan^{-1}(2)-\tan^{-1}(0)+\tan^{-1}(3)-\tan^{-1}(1)+\cdots+\tan^{-1}(101)-\tan^{-1}(99)\end{aligned}$$

Which clearly telescopes to

$$\tan^{-1}(101)+\tan^{-1}(100)-\tan^{-1}(1)-\tan^{-1}(0)=\tan^{-1}(101)+\tan^{-1}(100)-\frac{\pi}{4}.$$

To extract the answer we can use tangent addition formula to get $\boxed{-\frac{5150}{4949}}$. □

15. **[54]** Acute triangle ABC has integer side lengths, and BC is prime. Let I be the intersection of the angle bisectors of B and C. Next, let J be the point such that $IB \perp JB$ and $IC \perp JC$. Given that IJ is also an integer, find the least possible perimeter of $\triangle ABC$.

Proposed by William Hua

Solution. $\boxed{21}$

Let $a=BC$, $b=CA$, and $c=AB$.

Notice that J is the A-excenter.

Let $\theta=\frac{\angle A}{2}$, and let R be the circumradius. Also, let M be the midpoint of arc BC that does not contain A. Then, $a=2R\sin(2\theta)=4R\sin(\theta)\cos(\theta)$. By the incenter-excenter lemma, $IJ=4R\sin(\theta)$.

Hence,

$$\cos(\theta)=\frac{a}{IJ}.$$

Because a is prime, then either the fraction is irreducible or $a \mid IJ$.

Case 1. $a \mid IJ$. Note that $\cos(\theta)<1$, so if $a \mid IJ$, then $IJ \geq 2a$. However, $IJ=2BM\geq 2a \implies BM \geq a$, which is not possible since ABC is acute.

Case 2. $a \nmid IJ$.

Note that

$$2\cos^2(\theta)-1=\cos(2\theta)=\frac{b^2+c^2-a^2}{2bc} \implies \cos(\theta)=\frac{1}{2}\sqrt{\frac{(b+c)^2-a^2}{bc}}.$$

Because the numerator of $\cos(\theta)$ is a, then $a \mid b+c$. Let $b+c=da$.

Now,

$$IJ=2\sqrt{\frac{bc}{d^2-1}}$$

is an integer.

If d is even, then IJ is even.

Note that $b+c=da$ and $bc=x^2(d^2-1)$ for some positive integer x.

Thus,

$$b=\frac{da\pm\sqrt{d^2a^2-4x^2(d^2-1)}}{2}=\frac{1}{2}da\pm\sqrt{\frac{1}{4}d^2a^2-x^2(d^2-1)}.$$

This comes down to Pell-like equations with a fundamental solution of $(a,x)=(2,1)$.

However, if $(a,x)=(2n,n)$, then ABC is degenerate.

Also, d should be small since the perimeter is $a(d+1)$. Trying $d=2$ gives a solution of $(a,x)=(7,4)$, so $b=6,8$, and c is the other number. This yields an acute triangle, and a is prime too. Trying larger d gives solutions that are too big (you can also just notice that the other prime a do not yield smaller solutions).

If d is odd, this time, say $b+c=da$ and $bc=\frac{x^2(d^2-1)}{4}$. Then,

$$b=\frac{da\pm\sqrt{d^2a^2-x^2(d^2-1)}}{2}.$$

Again, the fundamental solution $(a,x)=(1,1)$ results in a degenerate triangle (and so does $(a,x)=(n,n)$ in general). Also note that $d>1$. Then, trying $d=3$ gives the next best solution $(a,x)=(9,11)$, but a is not prime.

Further experimentation results in the conclusion that odd d does not give a better triangle. Note that we only need to check $a=2,3,5$ since we found that $a=7, d=2$ works previously, and if $a\geq 11$, then the perimeter is greater than 22.

Hence, the answer is $\boxed{21}$. □