Team Round

LMT Spring 2024

May 4, 2024

1. **[16]** How many positive divisors of 160 are not divisors of 36?
2. **[18]** On May 4th, Yoda wakes up at a random time between 7 AM and 8 AM. On that same day, his planet will get invaded at a random time between 7:35 AM and 7:45 AM. What is the probability that Yoda wakes up before the invasion?
3. **[20]** Find the sum of the solutions to the equation $(x+3)^3+(x+2)^2+(x+1)^1+(x+0)^0=0$.
4. **[22]** In quadrilateral $ABCD$, let $\angle ABD = 32^\circ$, $\angle CBD = 64^\circ$, $\angle CDB = 58^\circ$, and $\angle ADB = 74^\circ$. Find $\angle ACD$.
5. **[24]** Andy starts with the number 2^{2024}. He can perform each of the following operations as many times as needed in any order. However an operation can only be used if the result would be an integer.

   The operations are:

   - Take the square root of the number.
   - Multiply the number by $\frac{5}{2}$.
   - Take the base-10 logarithm of the number.

   Let m be the minimum achievable output after a series of operations. Find the minimum number of operations it takes Andy to produce m.
6. **[26]** Peter begins at the origin on the xy-plane with a device that allows him to move 1 unit in a straight line in any direction. After using the device 3 times, Peter ends up at $(1,0)$. Let S denote the set of all points Peter could have passed through along his trip. Find the area of S.
7. **[28]** Find the number of ways to fill in the cells of a 3×3 grid with $\{1,1,2,2,3,3,4,4,5\}$ such that there are exactly 5 cells with strictly greater values than their neighbors (cells that share an edge).
8. **[30]** Let $f(n)$ denote the least positive integer k such that k has exactly n positive integer divisors. Let

   $$N=\sum_{n=1}^{100} f(n).$$

   Find $\lfloor \log_2(N)\rfloor$.
9. **[32]** Evin and his friends often think about *The Game*. Starting at noon, Evin thinks about The Game every 3 minutes on the dot, Jerry thinks about the game every 4 minutes on the dot, and Billiam thinks about the game every 6 minutes on the dot. Whenever any one of them thinks about The Game, there is a 50-50 chance that they announce "I lost the game", which causes each other friend to think about the game as well. What is the expected number of minutes from noon until the next time that all three friends think about the game simultaneously? (This could happen either because a friend announced that they lost the game, or because all three friends privately think about the game at the same instant.)
10. **[34]** There are 2021 heads-up coins arranged in a circle, and Jelfin starts next to an arbitrary (heads-up) coin. For all positive integers i, after the ith second, Jelfin moves i coins in the clockwise direction and then flips the coin he's moved next to. After n seconds, Jelfin lands on a coin which is tails-up. Find the least possible value of n.

11. **[38]** In triangle ABC, let $AB = 3$, $BC = 4$, and $CA = 5$. Define the Sakupen Circle $(X, \triangle XYZ)$ of a vertex X with respect to $\triangle XYZ$, as the circle passing through X, the midpoint of XY, and the midpoint of XZ. Let M, N, and P be the midpoints of sides AB, BC, and CA, respectively. Let $J \neq A$ be the intersection of segment AN and (A, ABC), and let $H \neq N$ be the intersection of segment AN and $(B, \triangle ABC)$. Find the area of quadrilateral $JMHP$.

12. **[42]** Let x, y, and z be real numbers satisfying

$$x^2 + y^2 = 1 + 2yz - xy$$
$$z^2 = 2xz - xy - 1$$

Find the minimum possible positive value of x.

13. **[46]** Let $\tau(n)$ denote the number of positive divisors of n. Let $\tau_1(n) = \tau(n)$ and for $k \geq 2$ let

$$\tau_k(n) = \sum_{d|n} \tau_{k-1}(d)\tau\left(\frac{n}{d}\right).$$

Find $\tau_5(2024)$.

14. **[50]** Evaluate

$$\tan\left(\sum_{x=1}^{100} \arctan\left(\frac{2}{x^2}\right)\right).$$

15. **[54]** Acute triangle ABC has integer side lengths, and BC is prime. Let I be the intersection of the angle bisectors of B and C. Next, let J be the point such that $IB \perp JB$ and $IC \perp JC$. Given that IJ is also an integer, find the least possible perimeter of $\triangle ABC$.