Speed Round Solutions

LMT Spring 2024

May 4, 2024

1. **[6]** Let $a \odot b = a(a+b)$ for all positive integers a and b. What's the value of $(3 \odot 7) \odot (4 \odot 1)$?

   Proposed by Calvin Garces

   Solution. $\boxed{1500}$

   We compute $3 \odot 7 = 30$ and $4 \odot 1 = 20$, so the answer is $30 \odot 20 = \boxed{1500}$. □

2. **[6]** How many positive integers less than 24 are not divisible by 3 or 5?

   Proposed by Peter Bai

   Solution. $\boxed{13}$

   There are 7 integers divisible by 3 (3 through 21), 4 integers divisible by 5 (5 through 20), and 1 integer divisible by $\operatorname{lcm}(3,5) = 15$. By PIE, there are $7+4-1 = 10$ integers less than 24 that are divisible by 3 or 5. Our answer is thus $23-10 = \boxed{13}$. □

3. **[6]** The number $2024 = 2^3+3^3+4^3+5^3+6^3+7^3+8^3+9^3$ can be expressed as the sum of eight consecutive cubes. Find the next smallest integer that has the same property.

   Proposed by Samuel Tsui

   Solution. $\boxed{3016}$

   The answer is $3^3+4^3+5^3+6^3+7^3+8^3+9^3+10^3$ which $2024+10^3-2^3 = \boxed{3016}$.

   Due to unclear wording, the answer $1^3+2^3+3^3+4^3+5^3+6^3+7^3+8^3 = 1296$ was also accepted. □

4. **[6]** What is the second smallest positive integer with a perfect square number of divisors that is also a perfect square?

   Proposed by Muztaba Syed and Derek Zhao

   Solution. $\boxed{36}$

   1 works. All perfect squares must have an odd number of factors, so there must be 9 factors. $\boxed{36}$ is the smallest number with 9 factors. □

5. **[6]** Yoda begins harvesting kyber crystals at noon, at a constant rate of one kyber crystal per minute. Sidious is asleep, but wakes up each hour on the dot and steals half of Yoda's current number of kyber crystals. After how many minutes will Yoda first have 100 kyber crystals at once?

   Proposed by Jacob Xu

   Solution. $\boxed{175}$

   After 60 minutes, Yoda will have $\frac{60}{2} = 30$ kyber crystals. After 60 more minutes, Yoda will have $\frac{30+60}{2} = 45$ kyber crystals. After 55 more minutes Yoda will have reached 100 kyber crystals, so the answer is $60+60+55 = \boxed{175}$ minutes. □

6. **[6]** Derek draws a square $ABCD$ with side length 10. Let M be the midpoint of BC and let P be an arbitrary point on the boundary of the square. Find the maximum possible area of $\triangle AMP$.

   Proposed by William Hua

   Solution. $\boxed{50}$

   The point farthest away from AM on $ABCD$ is D, so this point maximizes the area. The area will be half the area of $ABCD$, which is $\frac{10^2}{2} = \boxed{50}$ □

7. **[6]** The date 12/31/23 is 123123 when the slashes are removed, which is a three-digit sequence repeated twice. Find the sum of the digits in the next such date. (Note that January 1st, 2024 is 01/01/24, which becomes 010124, not 1124.)

   Proposed by Samuel Tsui

   Solution. $\boxed{8}$

   The next year when this is possible is 30 because the date cannot start with a digit greater than 3. Checking this it fails. Trying 31 we can minimize the date by placing a 0, giving us an answer of 03/10/31 $\Rightarrow 0+3+1+0+3+1 = \boxed{8}$. □

8. **[6]** Consider the three arithmetic sequences: $\{a_1, a_2, \ldots\}$; $\{b_1, b_2, \ldots\}$; and $\{c_1, c_2, \ldots\}$. Define $d_n = a_n + b_{2n} + c_{n+3}$ for all positive integers n. Given that $d_1 = 5$ and $d_3 = 21$, find d_6.

   Proposed by Benjamin Yin

   Solution. $\boxed{45}$

   We see that d_n is a sum of three arithmetic sequences, which is also an arithmetic sequence. So, we can find d_6 from $d_1 + 8 \cdot 5 = 45$. □

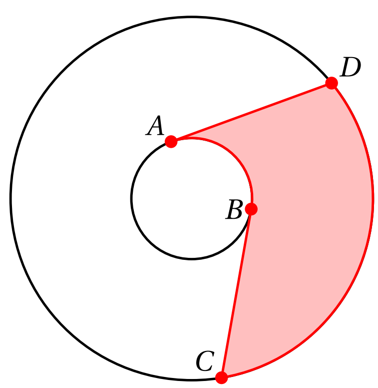
9. **[6]** For some real numbers a and b, the function $f(x) = x^2 + ax + b$ in x satisfies $f(3) - f(2) = 26$ and $f(0) = 0$. Evaluate $f(7)$.

   Proposed by Atticus Oliver

   Solution. $\boxed{196}$

   We can solve for a and b. The first equation tells us $5 + a = 26$, so $a = 21$. $f(0) = b$, so $b = 0$. Thus $f(x) = x^2 + 21x$, so the answer is $7^2 + 21 \cdot 7 = \boxed{196}$ □

10. **[6]** Circles ω_1 and ω_2 are concentric and have areas of 1 and 100 respectively. Points A and B lie on ω_1 and subtend a $120°$ arc. Points C and D lie on ω_2 such that AD and BC are tangent to ω_1 as shown below.



   Find the area of the region bounded by arc BA, segment AD, arc DC, and segment CB (the shaded region above).

   Proposed by Muztaba Syed

   Solution. $\boxed{33}$

   Consider the point F on ω_1 which is $120°$ away from A and B. Drawing a tangent from there we see the region between ω_2 and ω_1 is split into 3 congruent regions by symmetry. Thus the answer is $\frac{100-1}{3} = \boxed{33}$. □

11. **[6]** How many ways are there to rearrange the letters in the word $THEFORCE$ such that no two vowels are next to each other?

Proposed by Jonathan Liu

Solution. $\boxed{7200}$

Consider 3 vowels that are E's and O's and 5 distinct consonants T, H, F, R, C. There are $\binom{6}{3} \cdot 3 = 60$ ways to place the vowels and 5! ways to rearrange the consonants for a total of $\boxed{7200}$. □

12. **[6]** Jerry folds a flat square piece of paper towel in half five times, such that the resulting shape is a right isosceles triangle ABC with $AB = BC = 4$cm. However, Jerry spills apple juice on corner B such that all parts of the paper towel within 2cm of B are stained. When Jerry unfolds the towel, what fraction is stained?

Proposed by Samuel Wang

Solution. $\boxed{\frac{\pi}{8}}$

Note that when the paper is folded in half, the area of paper towel stained with apple juice is halved, as is the total area of paper towel. Thus, the proportion remains constant when Jerry folds the paper in half. So, the proportion remains constant when folding or unfolding the paper. This means that it is enough to find the proportion of paper stained with apple juice after 5 folds. The total area of the paper is 8cm^2 and the area stained with apple juice is $2^2 \cdot \pi \cdot \frac{90^\circ}{360^\circ} = \pi$. Thus, the proportion is $\boxed{\frac{\pi}{8}}$. □

13. **[6]** Muztaba has two ordered triples (r_1, r_2, r_3) and (c_1, c_2, c_3) of real numbers. In a 3×3 grid, he fills in the cell in row i and column j with $r_i \cdot c_j$. Some values in the grid are shown below:

		42
21	28	
	32	56

Find the sum of all values in the grid (including the ones not shown).

Proposed by Muztaba Syed

Solution. $\boxed{294}$

All the values can be calculated by using the ratios of rows/columns, but for efficiency we can also guess the values of (r_1, r_2, r_3) and (c_1, c_2, c_3). Indeed the 2nd row has elements divisible by 7, so $r_2 = 7$, meaning $c_1 = 3$ and $c_2 = 4$. This gives $r_3 = 8$, from which $c_3 = 7$ and $r_1 = 6$. The total sum is then

$$(r_1 + r_2 + r_3)(c_1 + c_2 + c_3) = (6+7+8)(3+4+7) = \boxed{294}.$$

□

14. **[6]** Let $n = \overline{a_l a_{l-1} \dots a_1 a_0}$, where each a_i is a digit. Consider the function $LMT(n)$ on positive integers:

$$LMT(n) = \begin{cases} n & 1 \le n \le 9 \\ LMT(a_l + a_{l-1} + \cdots + a_0) & n > 9. \end{cases}$$

Find the number of positive integers $m \le 1000$ satisfying $LMT(m) = 5$.

Proposed by Irene Choi

Solution. $\boxed{111}$

Let the sum of the digits of n_i be n_{i+1}. Then, noting that

$$n \equiv n_1 \equiv n_2 \equiv \dots \equiv n_i \equiv LMT(n)$$

in modulo 9, we have

If $9 \mid n$, $LMT(n) = 9$ and if $9 \nmid n$, $LMT(n)$ is the remainder of n when divided by 9.

So, for $LMT(n) = 5$, $n \equiv 5 \pmod 9$. Therefore, there are $\boxed{111}$ n's within the given range. □

15. **[6]** Consider regular heptagon $BCDEFGH$ centered at point A. Andrew calls a given triangle a *tringle* if it "doesn't contain A", that is, if A is not in the interior or on the boundary of the given triangle. If Andrew chooses 3 of the 8 points to be the vertices of a triangle, what's the probability that it's also a tringle?

Proposed by Zachary Perry

Solution. $\boxed{\frac{3}{8}}$

If the triangle is a tringle, point A cannot be one of the three chosen points. By inspection, the only triangles that are tringles are made up of three consecutive points, or four consecutive points with either of the two middle points removed. This means that the answer is $\frac{7+7+7}{\binom{8}{3}} = \boxed{\frac{3}{8}}$. □

16. **[6]** While running, Derman's speed is c meters per second, which slows to $\frac{c}{2}$ when jumping over a hurdle. Derman sets up 5 hurdles in the 60-meter race and 10 hurdles in the 110-meter race and runs times of 9 seconds and 17 seconds respectively. If Derman starts jumping d meters before each hurdle and lands d meters after, find $c+d$. (Assume that the d meters before and after each hurdle do not overlap and do not extend outside of the designated race distance.)

Proposed by Jason Yang

Solution. $\boxed{13}$

In the first race Derman runs $10d$ meters of hurdles and $60-10d$ meters of non-hurdles. With similar logic on the second race we get

$$\frac{60+10d}{c} = 9$$

and

$$\frac{110+20d}{c} = 17$$

Solving we get $c = 10$ and $d = 3$, so the answer is $3+10 = \boxed{13}$. □

17. **[6]** Let the set of *modern primes* be the set of primes, as well as their negatives. As a result, many integers such as $15 = 3\cdot 5 = (-3)\cdot(-5)$ do not have a unique modern prime factorization. How many integers with absolute value less than 100 have a unique modern prime factorization? (Do not include -1, 0, or 1 in your count.)

Proposed by Christopher Cheng

Solution. $\boxed{54}$

Clearly normal primes will work, and we see that negative primes as well as negative squares of primes also work. For numbers with more than 1 distinct prime divisors we can arbitrarily negate them to get multiple ways. Positive prime squares also have multiple ways, and negative prime cubes have multiple ways. In total this gives us an answer of $25+25+4 = \boxed{54}$. □

18. **[6]** In quadrilateral $ABCD$, let point E be the intersection of AB and CD closer to B than A, and let point F be the intersection of AD and BC, closer to D than A. Given that $\angle E = 20°$, $\angle F = 24°$, $\angle BAC = 35°$, and $\angle CAD = 33°$, find $\angle ABD$.

Proposed by Samuel Tsui

Solution. $\boxed{55}$

From triangle AED we get that $\angle ADC = 180-35-33-20 = 92°$ and from triangle AFB we get that $\angle ABC = 180-35-33-24 = 88°$. This means $ABCD$ is cyclic. Finally from triangle ADC we get $\angle ACD = \angle ABD = 180-33-92 = \boxed{55°}$. □

19. **[6]** Find the sum of all real solutions to $3x^2\log_2(x) - 3x\log_2(x^2) + \log_2(x^3) = 4x^2 - 8x + 4$.

Proposed by William Hua

Solution. $\boxed{1+2\sqrt[3]{2}}$

This is equivalent to

$$4(x-1)^2 = 3x^2\log_2(x) - 6x\log_2(x) + 3\log_2(x) = 3(x-1)^2\log_2(x)$$

This means that $x = 1$ or $\log_2(x) = \frac{4}{3}$. This gives us an answer of $\boxed{1+2\sqrt[3]{2}}$ □

20. **[6]** The letters of $XMASCOOKIE$ are placed into a bag and scrambled. Derek draws three random letters, returns a letter of his choice, and then draws three more letters. What is the probability that he can spell $XOOKS$ with his letters, assuming he optimally pursues this goal?

Proposed by William Hua

Solution. $\boxed{\frac{1}{112}}$

Call a letter "desirable" if it is X, O, K, or S.

We can treat the two Os as distinct, for we need both Os in $XOOKS$.

Then, Derek needs to draw at least two desirable letters at the start.

If Derek draws exactly two desirable letters at the start, then he discards the undesirable letter. This contributes a probability of $\frac{\binom{5}{2}\cdot 5}{\binom{10}{3}\binom{8}{3}}$ of winning from this case.

The probability that Derek gets three desirable letters and then wins is $\frac{\binom{5}{3}}{\binom{10}{3}\binom{8}{3}}$.

In total this gives us and answer of $\boxed{\frac{1}{112}}$. □

21. **[6]** Distinct primes p_1, p_2, p_3, and p_4 satisfy

$$p_1^2 + p_2^2 + p_3^2 = 46p_4^2$$

Find the sum of all possible values of $p_1p_2p_3p_4$.

Proposed by Samuel Tsui

Solution. $\boxed{1920}$

Notice the right hand side of the equation is even so one of the primes on the left must be 2. Additionally since every prime squared mod 3 is 1 except for 3 itself, $p_4 = 3$. Testing gives that the other two primes are either 7 and 19 or 11 and 17 thus the answer is $2\cdot3\cdot7\cdot19+2\cdot3\cdot11\cdot17 = 798+1122 = \boxed{1920}$. □

22. **[6]** Define the sequence $\{a_i\}$ as follows: let $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1}a_{n-2} + a_{n-1} + a_{n-2}$ for $n \geq 2$. Find $a_{14} \pmod{1000}$.

Proposed by Samuel Tsui

Solution. $\boxed{23}$

Note the expression can be rewritten as $a_n + 1 = (a_{n-1}+1)(a_{n-2}+1)$. Thus if we define $b_n = a_n + 1$ then $b_n = b_{n-1}b_{n-2}$. Since $b_0 = b_1 = 2$ this gives $b_{14} = 2^{610}$ (the exponents of 2 form the Fibonacci sequence). Now we want to find the value of this expression mod 125 and 8. $2^{610} \equiv 2^{10} \equiv 24 \pmod{125}$, and $2^{610} \equiv 0 \pmod{8}$. Thus $b_{14} \equiv 24 \pmod{1000}$, so the answer is $\boxed{23}$. □

23. **[6]** Hexagon $ABCDEF$ with $AB = 7$, $AC = 10$, $AD = 12$, $AE = 9$, and $AF = 4$ has parallel and congruent pairs of opposite sides. Point X lies in the interior of the hexagon. Find the least possible value of

$$AX^2 + BX^2 + CX^2 + DX^2 + EX^2 + FX^2.$$

Proposed by Muztaba Syed

Solution. $\boxed{174}$

First we will characterize the point X. Note that $ABDE$, $ACDF$, and $BCEF$ are all parallelograms. This implies that AD and BE share a midpoint, and likewise this midpoint is also the midpoint of CF. Call this point P.

If we just want to minimize $AX^2 + DX^2$, we will let X be the midpoint of AD. The same applies for $BX^2 + EX^2$ and $CX^2 + FX^2$, and because they all share a midpoint this point is optimal. Thus $X = P$.

This means our final answer is $\frac{1}{2}\cdot(AD^2 + BE^2 + CF^2)$. To get the answer we can apply the Parallelogram Law on $ABDE$ and $AFDC$. This gives us

$$AD^2 + BE^2 = 2(7^2 + 9^2)$$

and

$$AD^2 + CF^2 = 2(4^2 + 10^2)$$

Adding these equations and subtracting $AD^2 = 12^2$ gives us a final answer of

$$7^2 + 10^2 + 9^2 + 4^2 - \frac{12^2}{2} = \boxed{174}$$

□

24. **[6]** Let

$$N = \frac{1}{300} + \frac{1}{301} + \cdots + \frac{1}{362}.$$

Evaluate $\lfloor 100N \rfloor$.

Proposed by Adam Ge

Solution. $\boxed{19}$

Since

$$\begin{aligned} N &= \frac{1}{300} + \cdots + \frac{1}{329} + \frac{1}{330} + \cdots + \frac{1}{362} \\ &< \frac{1}{300} + \cdots + \frac{1}{300} + \frac{1}{330} + \cdots + \frac{1}{330} \\ &= 0.2 \end{aligned}$$

and

$$\begin{aligned} N &= \frac{1}{300} + \frac{1}{301} + \cdots + \frac{1}{361} + \frac{1}{362} \\ &= \left(\frac{1}{300} + \frac{1}{362}\right) + \left(\frac{1}{301} + \frac{1}{361}\right) + \cdots + \left(\frac{1}{330} + \frac{1}{332}\right) + \frac{1}{331} \quad > \underbrace{\frac{2}{331} + \frac{2}{331} + \cdots + \frac{2}{331}}_{31 \text{ times}} + \frac{1}{331} = \frac{63}{331} \quad > 0.19, \end{aligned}$$

N is between 0.19 and 0.20. Therefore the answer is $\boxed{19}$. □

25. **[6]** Consider triangle $A_0B_0C_0$ with $A_0B_0 = 13$, $B_0C_0 = 14$, and $C_0A_0 = 15$. For every nonnegative integer n, let ω_n denote the incircle of $A_nB_nC_n$ and denote $A_{n+1}B_{n+1}C_{n+1}$ recursively to be the triangle with circumcircle ω_n such that $\overline{A_{n+1}B_{n+1}} \parallel \overline{A_nB_n}$, $\overline{B_{n+1}C_{n+1}} \parallel \overline{B_nC_n}$, $\overline{C_{n+1}A_{n+1}} \parallel \overline{C_nA_n}$, and A_{n+1} lies between lines $B_{n+1}C_{n+1}$ and B_nC_n. Suppose that point P lies inside ω_n for all positive integers n. If O is the circumcenter of $A_0B_0C_0$, find PO.

Proposed by Calvin Garces and Jerry Xu

Solution. $\boxed{\dfrac{65\sqrt{65}}{776}}$

By memorization, $R = \frac{65}{8}$ and $r = 4$ where R and r are the circumradius and inradius of $A_0B_0C_0$, respectively. Let I be the incenter of $A_0B_0C_0$. By $OI^2 = R(R-2r)$ (Euler's Inequality), we see that $OI = \frac{\sqrt{65}}{8}$. Notice that P is the point of negative homothety that sends O to I. Therefore,

$$\frac{PO}{PI} = \frac{65}{8\cdot 4} = \frac{65}{32}.$$

Therefore, $PO = \frac{65}{97}\cdot\frac{\sqrt{65}}{8} = \boxed{\dfrac{65\sqrt{65}}{776}}$. □