Speed Round

LMT Spring 2024

May 4, 2024

1. **[6]** Let $a \odot b = a(a+b)$ for all positive integers a and b. What's the value of $(3\odot 7)\odot(4\odot 1)$?

2. **[6]** How many positive integers less than 24 are not divisible by 3 or 5?

3. **[6]** The number $2024 = 2^3+3^3+4^3+5^3+6^3+7^3+8^3+9^3$ can be expressed as the sum of eight consecutive cubes. Find the next smallest integer that has the same property.

4. **[6]** What is the second smallest positive integer with a perfect square number of divisors that is also a perfect square?

5. **[6]** Yoda begins harvesting kyber crystals at noon, at a constant rate of one kyber crystal per minute. Sidious is asleep, but wakes up each hour on the dot and steals half of Yoda's current number of kyber crystals. After how many minutes will Yoda first have 100 kyber crystals at once?

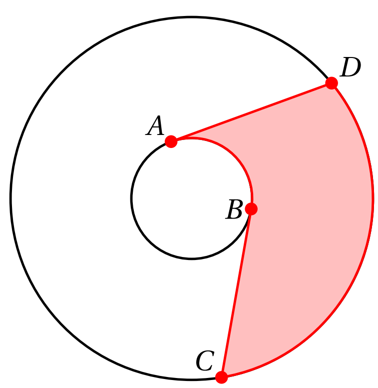
6. **[6]** Derek draws a square $ABCD$ with side length 10. Let M be the midpoint of BC and let P be an arbitrary point on the boundary of the square. Find the maximum possible area of $\triangle AMP$.

7. **[6]** The date 12/31/23 is 123123 when the slashes are removed, which is a three-digit sequence repeated twice. Find the sum of the digits in the next such date. (Note that January 1st, 2024 is 01/01/24, which becomes 010124, not 1124.)

8. **[6]** Consider the three arithmetic sequences: $\{a_1, a_2, \dots\}$; $\{b_1, b_2, \dots\}$; and $\{c_1, c_2, \dots\}$. Define $d_n = a_n + b_{2n} + c_{n+3}$ for all positive integers n. Given that $d_1 = 5$ and $d_3 = 21$, find d_6.

9. **[6]** For some real numbers a and b, the function $f(x) = x^2 + ax + b$ in x satisfies $f(3) - f(2) = 26$ and $f(0) = 0$. Evaluate $f(7)$.

10. **[6]** Circles ω_1 and ω_2 are concentric and have areas of 1 and 100 respectively. Points A and B lie on ω_1 and subtend a 120° arc. Points C and D lie on ω_2 such that AD and BC are tangent to ω_1 as shown below.



Find the area of the region bounded by arc BA, segment AD, arc DC, and segment CB (the shaded region above).

11. **[6]** How many ways are there to rearrange the letters in the word $THEFORCE$ such that no two vowels are next to each other?

12. **[6]** Jerry folds a flat square piece of paper towel in half five times, such that the resulting shape is a right isosceles triangle ABC with $AB = BC = 4$cm. However, Jerry spills apple juice on corner B such that all parts of the paper towel within 2cm of B are stained. When Jerry unfolds the towel, what fraction is stained?

13. **[6]** Muztaba has two ordered triples (r_1, r_2, r_3) and (c_1, c_2, c_3) of real numbers. In a 3×3 grid, he fills in the cell in row i and column j with $r_i \cdot c_j$. Some values in the grid are shown below:

		42
21	28	
	32	56

Find the sum of all values in the grid (including the ones not shown).

14. **[6]** Let $n = \overline{a_l a_{l-1} \dots a_1 a_0}$, where each a_i is a digit. Consider the function $LMT(n)$ on positive integers:

$$LMT(n) = \begin{cases} n & 1 \le n \le 9 \\ LMT(a_l + a_{l-1} + \dots + a_0) & n > 9. \end{cases}$$

Find the number of positive integers $m \le 1000$ satisfying $LMT(m) = 5$.

15. **[6]** Consider regular heptagon $BCDEFGH$ centered at point A. Andrew calls a given triangle a *tringle* if it "doesn't contain A", that is, if A is not in the interior or on the boundary of the given triangle. If Andrew chooses 3 of the 8 points to be the vertices of a triangle, what's the probability that it's also a tringle?

16. **[6]** While running, Derman's speed is c meters per second, which slows to $\frac{c}{2}$ when jumping over a hurdle. Derman sets up 5 hurdles in the 60-meter race and 10 hurdles in the 110-meter race and runs times of 9 seconds and 17 seconds respectively. If Derman starts jumping d meters before each hurdle and lands d meters after, find $c + d$. (Assume that the d meters before and after each hurdle do not overlap and do not extend outside of the designated race distance.)

17. **[6]** Let the set of *modern primes* be the set of primes, as well as their negatives. As a result, many integers such as $15 = 3 \cdot 5 = (-3) \cdot (-5)$ do not have a unique modern prime factorization. How many integers with absolute value less than 100 have a unique modern prime factorization? (Do not include -1, 0, or 1 in your count.)

18. **[6]** In quadrilateral $ABCD$, let point E be the intersection of AB and CD closer to B than A, and let point F be the intersection of AD and BC, closer to D than A. Given that $\angle E = 20°$, $\angle F = 24°$, $\angle BAC = 35°$, and $\angle CAD = 33°$, find $\angle ABD$.

19. **[6]** Find the sum of all real solutions to $3x^2 \log_2(x) - 3x \log_2(x^2) + \log_2(x^3) = 4x^2 - 8x + 4$.

20. **[6]** The letters of $XMASCOOKIE$ are placed into a bag and scrambled. Derek draws three random letters, returns a letter of his choice, and then draws three more letters. What is the probability that he can spell $XOOKS$ with his letters, assuming he optimally pursues this goal?

21. **[6]** Distinct primes p_1, p_2, p_3, and p_4 satisfy

$$p_1^2 + p_2^2 + p_3^2 = 46p_4^2$$

Find the sum of all possible values of $p_1 p_2 p_3 p_4$.

22. **[6]** Define the sequence $\{a_i\}$ as follows: let $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1}a_{n-2} + a_{n-1} + a_{n-2}$ for $n \ge 2$. Find a_{14} (mod 1000).

23. **[6]** Hexagon $ABCDEF$ with $AB = 7$, $AC = 10$, $AD = 12$, $AE = 9$, and $AF = 4$ has parallel and congruent pairs of opposite sides. Point X lies in the interior of the hexagon. Find the least possible value of

$$AX^2 + BX^2 + CX^2 + DX^2 + EX^2 + FX^2.$$

24. **[6]** Let

$$N = \frac{1}{300} + \frac{1}{301} + \dots + \frac{1}{362}.$$

Evaluate $\lfloor 100N \rfloor$.

25. **[6]** Consider triangle $A_0B_0C_0$ with $A_0B_0 = 13$, $B_0C_0 = 14$, and $C_0A_0 = 15$. For every nonnegative integer n, let ω_n denote the incircle of $A_nB_nC_n$ and denote $A_{n+1}B_{n+1}C_{n+1}$ recursively to be the triangle with circumcircle ω_n such that $\overline{A_{n+1}B_{n+1}} \parallel \overline{A_nB_n}$, $\overline{B_{n+1}C_{n+1}} \parallel \overline{B_nC_n}$, $\overline{C_{n+1}A_{n+1}} \parallel \overline{C_nA_n}$, and A_{n+1} lies between lines $B_{n+1}C_{n+1}$ and B_nC_n. Suppose that point P lies inside ω_n for all positive integers n. If O is the circumcenter of $A_0B_0C_0$, find PO.