LMT Spring 2024 Guts Round Solutions- Part 1

Team Name: ______________

______ 1. **[12]** Given that $\frac{2^{0-2\cdot 4}}{2\cdot 0!+2^{-4}}$ can be expressed as $\frac{a}{b}$ where a and b are relatively prime integers, calculate $20a+\frac{b}{24}$.

Proposed by Atticus Oliver

Solution. $\boxed{42}$

Evaluate the expression to get $\frac{\frac{1}{256}}{\frac{33}{16}} = \frac{1}{528}$, so $20a+\frac{b}{24} = 20(1)+\frac{528}{24} = 20+22 = \boxed{42}$. □

______ 2. **[12]** How many permutations $(a_1, a_2, \dots, a_{10})$ of $(1,2,\dots,10)$ are there such that $\left\lfloor \frac{a_k}{k} \right\rfloor$ is odd for all $1 \le k \le 10$?

Proposed by Muztaba Syed

Solution. $\boxed{1}$

Note that if $a_k < k$ this doesn't work (since $\left\lfloor \frac{a_k}{k} \right\rfloor = 0$), so $a_k \ge k$ for all $1 \le k \le 10$. But this just means that $a_1 = 1$, so $a_2 = 2$, and thus $a_k = k$. So there is only $\boxed{1}$ working permutation. □

______ 3. **[12]** In pentagon $DZHAO$, DZ is parallel to HA, ZH is perpendicular to HA, and $\angle AOD$ is 90°. Given that $DZ = HA = 10$, $ZH = 8$, and $AO = DO$, find $[DZHAO]$.

Proposed by Calvin Garces

Solution. $\boxed{96}$

Split the pentagon into a rectangle $DZHA$ and isosceles triangle AOD. We see the area is $10\cdot 8+4\cdot 4 = \boxed{96}$. □

LMT Spring 2024 Guts Round Solutions- Part 2

Team Name: ______________

______ 4. **[15]** Given that $x - \lfloor y \rfloor = \frac{107}{7}$ and $y - \lceil x \rceil = -\frac{139}{9}$, find $x-y$.

Proposed by Atticus Oliver

Solution. $\boxed{\frac{928}{63}}$

Notice that $y - \lceil x \rceil = -\frac{144}{9} + \frac{5}{9} = -16 + \frac{5}{9}$. Since $\lceil x \rceil$ is always an integer, it must be that y is equal to some integer plus $\frac{5}{9}$. Therefore $\lfloor y \rfloor = y - \frac{5}{9}$. Substituting this into the first equation gives $x - \left(y - \frac{5}{9}\right) = \frac{107}{7}$, so $x - y = \frac{107}{7} - \frac{5}{9}$, so $x - y = \boxed{\frac{928}{63}}$. □

______ 5. **[15]** Derek's bike is large: The front wheel has radius 6 feet and the back wheel has radius 8 feet. He takes a ride, stopping when the front wheel has revolved exactly 5 more times than the back wheel. How far did Derek travel, in feet?

Proposed by Muztaba Syed and Derek Zhao

Solution. $\boxed{240\pi}$

Notice that the front wheel revolves $d(1/(12\pi) - 1/(16\pi)) = d/(48\pi)$ more times. Therefore, $d = \boxed{240\pi}$. □

6. **[15]** Sam has two fair dice: One die has faces labelled with $\{0,1,2,3,4,5\}$ and the other has faces labelled with $\{0,6,12,18,24,30\}$. Eddie has two fair spinners: One spinner has sections labelled with $\{0,1,2,3,4,5\}$ and the other has sections labelled with $\{0,1,2,3,4,5,6,7\}$. Sam rolls his two dice and Eddie spins his two spinners. What is the probability that the sum of Sam's results equals the product of Eddie's results?

Proposed by Chris Cheng

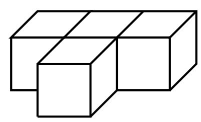
Solution. $\boxed{\frac{1}{36}}$

Notice Sam's range covers each number from 0 to 35 inclusive exactly once. Every number from Eddie's range is in Sam's range, therefore whatever Eddie gets, Sam has a $\boxed{\frac{1}{36}}$ chance of matching it. □

LMT Spring 2024 Guts Round Solutions- Part 3

Team Name:

7. **[18]** Jacob has four 6-sided dice in the shape of unit cubes. Each die has its 6 faces labelled with $\{1,2,3,4,5,6\}$ such that all pairs of opposite sides sum to 7. He arranges these dice so that they form a T-shape as shown below. What is the largest possible sum of numbers on the surface of the shape? (The bottom is included on the surface.)



Proposed by Peter Bai and Jacob Xu

Solution. $\boxed{73}$

Some sides are covered up, but disregarding this the sum is $7 \cdot 3 \cdot 4 = 84$. For the cube in the middle it loses a total of 7 and the last side loses at least 1. The other three cubes lose at least 1. In total the answer is $84 - 7 - 1 - 1 - 1 - 1 = \boxed{73}$ □

8. **[18]** Let $a_0 = 1007$, and for all positive integers n, let a_n be the result when we insert the digit 1 into the tens place of a_{n-1}. (Thus, $a_1 = 10017$, $a_2 = 100117$, and so on.) Find $\gcd(a_{2023}, a_{2024}, a_{2025})$.

Proposed by Samuel Wang

Solution. $\boxed{53}$

$a_{n+1} = 10a_n - 53$. Checking, $a_0 = 53 \cdot 19$. Thus, for all n, $53 | a_n$. $\gcd(a_n, 10a_n - 53) = \gcd(a_n, a_n - 53) = \gcd(53, a_n) = 53$. The answer is thus $\boxed{53}$ □

9. **[18]** Adam throws a dart that lands uniformly at random on a dartboard. The dartboard is in the shape of two overlapping regular hexagons $TOPHER$ and $PICKLE$, where T lies outside of $PICKLE$. What is the probability that the dart lands in quadrilateral $ROCK$?

Proposed by Adam Ge

Solution. $\boxed{\frac{16}{23}}$

WLOG, $RL = 2\sqrt{21}$. Using law of cosines on $\triangle REL$ and noting that $EL = RE\sqrt{3}$, we deduce that the side length of $TOPHER$ is $2\sqrt{3}$ and the side length of $PICKLE$ is 6. Since $RE \perp EP$ and $KE \perp EP$,

R, E, and K are collinear; similarly, O, P, and C are collinear. This implies that quadrilateral $ROCK$ is a rectangle.

$CK = 6$ and $RK = RE+EK = 2\sqrt{3}+6\sqrt{3} = 8\sqrt{3}$, so the area $ROCK$ is $48\sqrt{3}$. The area of the dartboard is equal to $[TOPHER]+[PICKLE]-[EHP] = 18\sqrt{3}+54\sqrt{3}-3\sqrt{3} = 69\sqrt{3}$, so the probability that the dart lands in $ROCK$ is $\frac{48\sqrt{3}}{69\sqrt{3}} = \boxed{\frac{16}{23}}$. □

LMT Spring 2024 Guts Round Solutions- Part 4

Team Name: ______________________________

__________ 10. **[21]** Find the coefficient of x^3 in the expansion of $(1+x+x^2+x^3+x^5)^6$.

Proposed by Jonathan Liu

Solution. $\boxed{56}$

Notice that the x^5 term does not affect the x^3 term in the expansion, so we need to find the coefficient of x^3 in $(1+x+x^2+x^3)^6$. By stars and bars this is equivalent to arranging 5 bars (splitting each of the 6 factors) and 3 bars (when the exponent increases), which gives us an answer of $\binom{5+3}{3} = \boxed{56}$. □

__________ 11. **[21]** Find the least positive integer n relatively prime to 14 such that $14^m + n$ is not prime for any nonnegative integer m.

Proposed by Evin Liang

Solution. $\boxed{11}$

First, note that $14^n + 11$ is divisible by 3 if n is even and 5 if n is odd. Now we check the smaller values of N:

$N = 1$ gives $14^0 + 1 = 2$ is prime $N = 3$ gives $14^1 + 3 = 17$ is prime $N = 5$ gives $14^1 + 5 = 19$ is prime $N = 9$ gives $14^1 + 9 = 23$ is prime

so $N = \boxed{11}$ is the smallest value that works. □

__________ 12. **[21]** In $\triangle ABC$ satisfying $AB = 15$, $BC = 20$, and $\angle ABC = 90°$, let D and E be points in the plane such that $DA = EC = 7$ and $\angle ADC = \angle CEA = 90°$. Find the minimum possible value of DE.

Proposed by Samuel Tsui

Solution. $\boxed{\frac{527}{25}}$

Observe from the fact that $\angle ADC = \angle CEA$, $ADEC$ is cyclic. Pythagorean theorem gives $AC = \sqrt{15^2+20^2} = 25$ and $EA = DC = \sqrt{25^2-7^2} = 24$. Additionally the minimum value of DE occurs when they are both on the same arc AC. Applying Ptolemy's Theorem we get $DE \cdot 25 + 7 \cdot 7 = 24 \cdot 24$ so $DE = \boxed{\frac{527}{25}}$. □

LMT Spring 2024 Guts Round Solutions- Part 5

Team Name: ______________________________

__________ 13. **[27]** Let S denote the curve $x^4y^4 - x^6 - y^6 + x^2y^2 = 0$ in the xy-coordinate plane, and let T denote the clockwise rotation of S about the origin by $45°$. At how many points do S and T intersect?

Proposed by Peter Bai

Solution. $\boxed{17}$

Notice that $x^4y^4 - x^6 - y^6 + x^2y^2$ factors to $(x^4 - y^2)(y^4 - x^2) = (x^2 - y)(x^2 + y)(y^2 - x)(y^2 + x)$. Since the graph of S is just when this is equal to 0, any point is in S if and only if it results in one of these 4 factors being zero. By setting each factor to 0, we get the following:

$x^2 - y = 0 \implies y = x^2$, $x^2 + y = 0 \implies y = -x^2$, $y^2 - x = 0 \implies x = y^2$, $y^2 + x = 0 \implies x = -y^2$.

As a result, the graph of S is the union of 4 parabolas: $y = x^2$, and its clockwise rotations about the origin by 90, 180, and 270 degrees.

Each new parabola intersects an old parabola at two points, one of which is the origin. Therefore, there are $1 + 4 \cdots 4 = \boxed{17}$ intersections. □

______ 14. **[27]** Let a, b, and c be the roots of the polynomial $3x^3 + 4x^2 + 3x + 4$.

Evaluate

$$\frac{a}{4a^2 + 3a + 4} + \frac{b}{4b^2 + 3b + 4} + \frac{c}{4c^2 + 3c + 4}.$$

Proposed by Jonathan Liu

Solution. $\boxed{\frac{23}{48}}$

Since a, b, c satisfy $3x^3 + 4x^2 + 3x + 4 = 0$, substitute $4x^2 + 3x + 4 = -3x^3$

The new fraction is $\frac{a}{-3a^3} + \frac{b}{-3b^3} + \frac{c}{-3c^3} = -\frac{1}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$ since a, b, c are nonzero.

It is well-known that switching coefficients results in reciprocal roots, so finding $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = r_1^2 + r_2^2 + r_3^2$, where r_1, r_2, r_3 are the roots of the equation $4x^3 + 3x^2 + 4x + 3 = 0$.

From Vieta or Newton's Sums, $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = r_1^2 + r_2^2 + r_3^2 = -\frac{23}{16}$, so the final answer is $-\frac{1}{3} \cdot -\frac{23}{16} = \boxed{\frac{23}{48}}$. □

______ 15. **[27]** Jiwu and 2024 other competitors have all received a perfect score on the Little Mini Tiny round designed for toddlers, so a tiebreaker is required. Each person's tiebreaker value is independently chosen at random from the real numbers between 0 and 1. Competitors are ranked from greatest to least tiebreaker value. However, each person also independently has $\frac{1}{2}$ chance of forgetting to fill out the tiebreaker, which results in them tie-ing for 2025th. What is Jiwu's expected placement?

Proposed by Samuel Wang

Solution. $\boxed{1266}$

We split into 2 cases:

Case 1: Jiwu forgets to fill out the tiebreaker. He then gets 2025-th with probability $\frac{1}{2}$.

Case 2: Jiwu fills out the tiebreaker. Note here that of the others, it is equally likely that x people forget to fill out the TB as $2024 - x$ others. The expected placement of the first situation is $\frac{2026-x}{2}$, and in the second is $\frac{x+2}{2}$, thus the expected value in this case $\frac{2028}{2\cdot 2} = 507$ with probability $\frac{1}{2}$.

The net expected value is thus $\frac{2025+507}{2} = \boxed{1266}$ □

LMT Spring 2024 Guts Round Solutions- Part 6

Team Name: ______

______ 16. **[33]** In hexagon $ABCDEF$ inscribed in a circle, $EF = FA = AB = BC$, and $CD = DE$. Suppose $FB = 14$ and $DE = 6$. Find the area of $ABCDEF$.

Proposed by Muztaba Syed

Solution. $\boxed{42+14\sqrt{58}}$

The equal sides imply that $14 = FB = AE$, and by symmetry A and D are diametrically opposite each other. This means $\angle AED = 90°$. By the Pythagorean theorem

$$AD = \sqrt{AE^2+ED^2} = \sqrt{14^2+6^2} = 2\sqrt{58}.$$

So the radius of the circle is $\sqrt{58}$.

Now rearrange the sides of the hexagon so that the sides with length 6 are opposite each other (this keeps the same area). Then we have two congruent isosceles trapezoids with bases 6 and $2\sqrt{58}$. These have height $\frac{14}{2} = 7$, so the answer is

$$2\cdot\frac{1}{2}\cdot 7(6+2\sqrt{58}) = \boxed{42+14\sqrt{58}}.$$

□

____________ 17. **[33]** Find the number of ordered quadruples (a,b,c,d) of nonnegative integers less than 12 such that $a+2b+3c+4d$ is a multiple of 5.

Proposed by Evin Liang

Solution. $\boxed{4148}$

Observe that $a+2b+3c+4d \equiv \overline{cdba}_{12} \mod 5$. Thus we want to find the number of multiples of 5 from 0 to 12^4-1 inclusive which is $1+\frac{12^4-1}{5} = \boxed{4148}$. □

____________ 18. **[33]** On the game show "haLl MonTy", there are three doors. Behind two of the doors is a pickle and behind the other door is Ella. You choose the right door, and the host has the option to reveal either the middle or left door. The host reveals the middle door, and behind it you see a pickle. You know that the host is one of the three following people chosen uniformly at random before your choice:

1. Jerry Xu: He knows where Ella is and will never reveal the door that she lies behind, otherwise he reveals a random door (left or middle).
2. Ben Yin: He does not know where Ella is and will reveal a random door (left or middle).
3. Evin Liang: He will not reveal the middle door unless Ella is behind the left door, in which case he will reveal the middle door.

Find the probability that there is a pickle behind the left door.

Proposed by Atticus Oliver

Solution. $\boxed{\frac{2}{7}}$

There are 3 possible hosts and 2 possible locations of Dr. Feng (right or left door), for a total of 6 possible car-host combinations. They are all analyzed exactly as below:

Jerry; Dr. Feng in right: He only chooses the middle door in this case half the time (as opposed to the left door), so this has weight $\frac{1}{2}$.

Jerry; Dr. Feng in left: He will always choose the middle door, so this has weight 1.

Ben; Dr. Feng in right: He will choose the middle door half the time (the other half he chooses the left door), so this has weight $\frac{1}{2}$.

Ben; Dr. Feng in left: Same as above, weight $\frac{1}{2}$ (nothing prevents Ben from opening the door with Dr. Feng, he just happened not to).

Evin; Dr. Feng in right: Impossible, Evin would have revealed the left door since he could.

Evin; Dr. Feng in left: He always reveals middle, so this has weight 1.

Of these, cases where the Dr. Feng is in the right have switching yield a pickle, so the answer is $\frac{\frac{1}{2}+\frac{1}{2}}{\frac{1}{2}+1+\frac{1}{2}+\frac{1}{2}+1} = \boxed{\frac{2}{7}}$. □

19. **[39]** Let $L > 0$ be the answer to problem 21. Jacob places $2L^2$ bishops and $2L^2$ rooks on a toroidal $2L \times 2L$ chessboard. Find the maximum possible number of ordered pairs (A, B) of two pieces on the board such that A attacks B but B does not attack A. (A toroidal chessboard is a normal chessboard, except it "loops around" at the end of the board. The first column and the last column are adjacent, as are the first and last rows. A rook attacks another piece if those two pieces are in the same row or column with no pieces between them. A bishop attacks another piece if those two pieces are in the same diagonal with no pieces between them.)

Proposed by Muztaba Syed

Solution. $\boxed{588}$

For a given L the answer is $12L^2$.

First A and B must be different types (rook and a bishop) for this to happen. Because our entire grid is full, they also must be in cells which share at least one vertex. We can double count the number of such pairs at each vertex of the grid. If a rook attacks a bishop they share a side, and this is counted at two vertices. If a bishop attacks a rook they share exactly one vertex. Thus when we are counting at each vertex sharing a side only counts as "half" since it is counted twice.

At a given vertex there are 4 pairs of cells which share an edge and 2 pairs which share only a vertex. We see by inspection that the maximum possible contribution of this vertex is when there are two adjacent rooks and two adjacent bishops, which contributes $\frac{1}{2} + \frac{1}{2} + 1 + 1 = 3$. Since there are $4L^2$ vertices we can have at most $3 \cdot 4L^2 = \boxed{12L^2}$ such pairs.

We can easily construct this by alternating rows of bishops and rows of rooks. This clearly works as each vertex has 2 adjacent rooks and two adjacent bishops. □

20. **[39]** Let M be the answer to problem 19. Let $DZHAO$ be a regular pentagon, and let R be the reflection of D over line HA. Given that $[ZHAO] = M$, evaluate $[DZHAO] + [ORZ]$.

Proposed by Muztaba Syed

Solution. $\boxed{1764}$

First we see that O, A, R are collinear (and similarly Z, H, R) because $\angle HAR = 72°$. Let the area of ZOD be a and ZAD be b. Then we see $[ZHAO] = a + b$, $[DZHAO] = 2a + b$, and $[ORZ] = 2b + a$. So our answer is $3a + 3b = 3(a + b) = 3[ZHAO] = \boxed{3M}$. □

21. **[39]** Let T be the answer to problem 20. Find the number of integers k between 1 and $\frac{T}{6}$ inclusive such that $\frac{3^k-1}{301}$ is an integer.

Proposed by Muztaba Syed and Sam Wang

Solution. $\boxed{7}$

We can factor $301 = 7 \cdot 43$. From $7 \mid 3^k - 1$ we get that $6 \mid k$. For $43 \mid 3^k - 1$, we see that the order of 3 mod 43 must be a divisor of 42, and from earlier we know $6 \mid k$. Thus it is either 6 or 42. We can easily verify that $43 \nmid 3^6 - 1$, so this means $42 \mid k$. There are $\left\lfloor \frac{T}{42 \cdot 6} \right\rfloor$ multiples of 42 in the given range.

Extracting answers for the cyclic round: We have $M = 12L^2$ and $T = 3M$, meaning $T = 36L^2$. Then we have $L = \left\lfloor \frac{L^2}{7} \right\rfloor$, from which we get $L = 7$. Putting this together we have $\boxed{L = 7}$, $\boxed{M = 588}$, and $\boxed{T = 1764}$. □

____________ 22. **[45]** Evaluate

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)2^{2n}}.$$

Proposed by Jerry Xu

Solution. $\boxed{2}$

Remember that $\frac{1}{n+1}\binom{2n}{n} = C_n$. Consider the following problem: You start at 0 on the number line. Each second, you flip a coin. If you flip a heads, you move 1 to the right. If you flip a tails, you move 1 to the left. What is the probability that you reach 1?

In order to reach 1, we must have a sequence of Ts and Hs with one more H than T. Suppose the last flip in the sequence is the first time we hit 1. We must then have that at no point in the sequence does the number of Hs exceed the number of Ts, for otherwise we would have hit 1. In addition, it is evident that the last flip in the sequence must be H: otherwise, the last flip would've been T and that would imply that we moved from 2 to 1. However, since we started from 0, that implies that we already reached 1. Say there are n Ts: the first n Ts and the first n Hs can be arranged in C_n ways, and the last H must go at the end: thus, the probability of hitting 1 after n Ts is $\frac{C_n}{2^{2n+1}}$. Summing this over all $n \in N^0$ gives the probability of ever reaching 1, which is obviously 1. Thus, our answer is $2 \cdot 1 = \boxed{2}$. □

____________ 23. **[45]** Evaluate

$$(\tan 5^\circ + \tan 85^\circ)(\tan 15^\circ + \tan 75^\circ)(\tan 25^\circ + \tan 65^\circ)(\tan 35^\circ + \tan 55^\circ).$$

Proposed by Evin Liang

Solution. $\boxed{256}$

Note that $\tan x + \tan(90^\circ - x) = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ which simplifies to $\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{2}{\sin 2x}$. Therefore, the big expression equals $\frac{16}{\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ}$. Since $\sin 30^\circ = \frac{1}{2}$ and $\sin x = \cos(90^\circ - x)$, this equals $\frac{32}{\cos 20^\circ \cos 40^\circ \cos 80^\circ}$. Note that if we multiply $\cos 20^\circ \cos 40^\circ \cos 80^\circ$ by $\sin 20^\circ$, it simplifies to $\frac{1}{2}\sin 40^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{4}\sin 80^\circ \cos 80^\circ = \frac{1}{8}\sin 160^\circ = \frac{1}{8}\sin 20^\circ$. Thus, $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$, so the answer is $\boxed{256}$. □

____________ 24. **[45]** In convex cyclic quadrilateral $ABCD$, let P be the foot of the altitude from D to BC, Q be the foot of the altitude from D to AC, and R be the foot of the altitude from A to BC. Let $S \neq D$ be the intersection of $(ABCD)$ with line $\overline{DP}$. Suppose lines $\overline{PQ}$ and $\overline{AR}$ intersect at X. Given $AR = 5$, $AS = 11$, and $PR = 4$, find the length of AX.

Proposed by Jerry Xu

Solution. $\boxed{\sqrt{105} - 5}$

$DPQC$ is cyclic as $\measuredangle DQC = \measuredangle DPC = 90^\circ$, so $\measuredangle ASD = \measuredangle ACD = \measuredangle QPD$, which implies that $\overline{AK} \parallel \overline{PQ}$. Recall that $\overline{PQ}$ is the Simson line of ABC with respect to D, so let T be the foot of the altitude from D to AB: it follows that P, Q, T are collinear.

Let $\overline{AR} \cap \overline{PQ} = X$. Then, note that $\overline{AR} \parallel \overline{DP}$, so $XASP$ is a parallelogram. It reduces to find PS. Note that $AS^2 = PR^2 + (AR + PS)^2$, so $121 = 16 + (5 + PS)^2 \Rightarrow PS = \boxed{\sqrt{105} - 5}$. □

LMT Spring 2024 Guts Round Solutions- Part 9

Team Name: ____________________

______ 25. **[30]** Submit one, two, or three positive integers. Let n be the number of integers you submitted to this question and let N be the total number of integers submitted across all teams. Let d be the least difference between N and one of your integers. You will receive $\left\lfloor \frac{30}{n(d^2+1)} \right\rfloor$ points.

Proposed by Aidan Duncan

Solution. $\boxed{}$

□

______ 26. **[30]** Let N be the number of obtuse triangles with integer side lengths that are at most 2024. Estimate N. Submit an integer. If you submit X, you will receive $\max\left(\left\lfloor 30 \cdot \min\left(\frac{N}{X}, \frac{X}{N}\right)\right\rfloor, 0\right)$ points.

Proposed by Evin Liang

Solution. $\boxed{282239670}$

Python program:

□

______ 27. **[30]** For all positive integers n, let $P(n)$'s value be the product of the digits of n (in base 10). Define $f(x) = \frac{P(x)}{x}$. Let M be the median of $\{f(1), f(2), \ldots, f(2024)\}$. If you submit X, you will recieve $\max(\lfloor 30 \cdot (1 - |X - M| \cdot 20) \rfloor, 0)$ points

Proposed by Aidan Duncan

Solution. $\boxed{126003/2100094 \approx 0.0599987429}$

program

□