LMT Spring 2024 Guts Round - Part 1

Team Name: _____

_____ 1. **[12]** Given that $\frac{2^{0-2\cdot 4}}{2\cdot 0!+2^{-4}}$ can be expressed as $\frac{a}{b}$ where a and b are relatively prime integers, calculate $20a+\frac{b}{24}$.

_____ 2. **[12]** How many permutations $(a_1,a_2,\ldots,a_{10})$ of $(1,2,\ldots,10)$ are there such that $\left\lfloor \frac{a_k}{k} \right\rfloor$ is odd for all $1\le k\le 10$?

_____ 3. **[12]** In pentagon $DZHAO$, DZ is parallel to HA, ZH is perpendicular to HA, and $\angle AOD$ is 90°. Given that $DZ=HA=10$, $ZH=8$, and $AO=DO$, find $[DZHAO]$.

LMT Spring 2024 Guts Round - Part 2

Team Name: _____

_____ 4. **[15]** Given that $x-\lfloor y\rfloor=\frac{107}{7}$ and $y-\lceil x\rceil=-\frac{139}{9}$, find $x-y$.

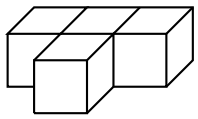
_____ 5. **[15]** Derek's bike is large: The front wheel has radius 6 feet and the back wheel has radius 8 feet. He takes a ride, stopping when the front wheel has revolved exactly 5 more times than the back wheel. How far did Derek travel, in feet?

_____ 6. **[15]** Sam has two fair dice: One die has faces labelled with $\{0,1,2,3,4,5\}$ and the other has faces labelled with $\{0,6,12,18,24,30\}$. Eddie has two fair spinners: One spinner has sections labelled with $\{0,1,2,3,4,5\}$ and the other has sections labelled with $\{0,1,2,3,4,5,6,7\}$. Sam rolls his two dice and Eddie spins his two spinners. What is the probability that the sum of Sam's results equals the product of Eddie's results?

LMT Spring 2024 Guts Round - Part 3

Team Name: _____

_____ 7. **[18]** Jacob has four 6-sided dice in the shape of unit cubes. Each die has its 6 faces labelled with $\{1,2,3,4,5,6\}$ such that all pairs of opposite sides sum to 7. He arranges these dice so that they form a T-shape as shown below. What is the largest possible sum of numbers on the surface of the shape? (The bottom is included on the surface.)



_____ 8. **[18]** Let $a_0=1007$, and for all positive integers n, let a_n be the result when we insert the digit 1 into the tens place of a_{n-1}. (Thus, $a_1=10017$, $a_2=100117$, and so on.) Find $\gcd(a_{2023},a_{2024},a_{2025})$.

_____ 9. **[18]** Adam throws a dart that lands uniformly at random on a dartboard. The dartboard is in the shape of two overlapping regular hexagons $TOPHER$ and $PICKLE$, where T lies outside of $PICKLE$. What is the probability that the dart lands in quadrilateral $ROCK$?

LMT Spring 2024 Guts Round - Part 4

Team Name: _____

_____ 10. **[21]** Find the coefficient of x^3 in the expansion of $(1+x+x^2+x^3+x^5)^6$.

_____ 11. **[21]** Find the least positive integer n relatively prime to 14 such that 14^m+n is not prime for any nonnegative integer m.

_____ 12. **[21]** In $\triangle ABC$ satisfying $AB=15$, $BC=20$, and $\angle ABC=90^\circ$, let D and E be points in the plane such that $DA=EC=7$ and $\angle ADC=\angle CEA=90^\circ$. Find the minimum possible value of DE.

LMT Spring 2024 Guts Round - Part 5

Team Name: _____

13. **[27]** Let S denote the curve $x^4y^4 - x^6 - y^6 + x^2y^2 = 0$ in the xy-coordinate plane, and let T denote the clockwise rotation of S about the origin by $45°$. At how many points do S and T intersect?

14. **[27]** Let a, b, and c be the roots of the polynomial $3x^3 + 4x^2 + 3x + 4$.

    Evaluate
    $$\frac{a}{4a^2+3a+4} + \frac{b}{4b^2+3b+4} + \frac{c}{4c^2+3c+4}.$$

15. **[27]** Jiwu and 2024 other competitors have all received a perfect score on the Little Mini Tiny round designed for toddlers, so a tiebreaker is required. Each person's tiebreaker value is independently chosen at random from the real numbers between 0 and 1. Competitors are ranked from greatest to least tiebreaker value. However, each person also independently has $\frac{1}{2}$ chance of forgetting to fill out the tiebreaker, which results in them tie-ing for 2025th. What is Jiwu's expected placement?

LMT Spring 2024 Guts Round - Part 6

Team Name: _____

16. **[33]** In hexagon $ABCDEF$ inscribed in a circle, $EF = FA = AB = BC$, and $CD = DE$. Suppose $FB = 14$ and $DE = 6$. Find the area of $ABCDEF$.

17. **[33]** Find the number of ordered quadruples (a, b, c, d) of nonnegative integers less than 12 such that $a + 2b + 3c + 4d$ is a multiple of 5.

18. **[33]** On the game show "haLl MonTy", there are three doors. Behind two of the doors is a pickle and behind the other door is Ella. You choose the right door, and the host has the option to reveal either the middle or left door. The host reveals the middle door, and behind it you see a pickle. You know that the host is one of the three following people chosen uniformly at random before your choice:

    1. Jerry Xu: He knows where Ella is and will never reveal the door that she lies behind, otherwise he reveals a random door (left or middle).
    2. Ben Yin: He does not know where Ella is and will reveal a random door (left or middle).
    3. Evin Liang: He will not reveal the middle door unless Ella is behind the left door, in which case he will reveal the middle door.

    Find the probability that there is a pickle behind the left door.

LMT Spring 2024 Guts Round - Part 7

Team Name: _____

19. **[39]** Let $L > 0$ be the answer to problem 21. Jacob places $2L^2$ bishops and $2L^2$ rooks on a toroidal $2L \times 2L$ chessboard. Find the maximum possible number of ordered pairs (A, B) of two pieces on the board such that A attacks B but B does not attack A. (A toroidal chessboard is a normal chessboard, except it "loops around" at the end of the board. The first column and the last column are adjacent, as are the first and last rows. A rook attacks another piece if those two pieces are in the same row or column with no pieces between them. A bishop attacks another piece if those two pieces are in the same diagonal with no pieces between them.)

20. **[39]** Let M be the answer to problem 19. Let $DZHAO$ be a regular pentagon, and let R be the reflection of D over line HA. Given that $[ZHAO] = M$, evaluate $[DZHAO] + [ORZ]$.

21. **[39]** Let T be the answer to problem 20. Find the number of integers k between 1 and $\frac{T}{6}$ inclusive such that $\frac{3^k-1}{301}$ is an integer.

LMT Spring 2024 Guts Round - Part 8

Team Name: _____

_____ 22. **[45]** Evaluate

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)2^{2n}}.$$

_____ 23. **[45]** Evaluate

$$(\tan 5^\circ + \tan 85^\circ)(\tan 15^\circ + \tan 75^\circ)(\tan 25^\circ + \tan 65^\circ)(\tan 35^\circ + \tan 55^\circ).$$

_____ 24. **[45]** In convex cyclic quadrilateral $ABCD$, let P be the foot of the altitude from D to BC, Q be the foot of the altitude from D to AC, and R be the foot of the altitude from A to BC. Let $S \neq D$ be the intersection of $(ABCD)$ with line $\overline{DP}$. Suppose lines $\overline{PQ}$ and $\overline{AR}$ intersect at X. Given $AR = 5$, $AS = 11$, and $PR = 4$, find the length of AX.

LMT Spring 2024 Guts Round - Part 9

Team Name: _____

_____ 25. **[30]** Submit one, two, or three positive integers. Let n be the number of integers you submitted to this question and let N be the total number of integers submitted across all teams. Let d be the least difference between N and one of your integers. You will receive $\left\lfloor \frac{30}{n(d^2+1)} \right\rfloor$ points.

_____ 26. **[30]** Let N be the number of obtuse triangles with integer side lengths that are at most 2024. Estimate N. Submit an integer. If you submit X, you will receive $\max\left(\left\lfloor 30 \cdot \min\left(\frac{N}{X}, \frac{X}{N}\right)\right\rfloor, 0\right)$ points.

_____ 27. **[30]** For all positive integers n, let $P(n)$'s value be the product of the digits of n (in base 10). Define $f(x) = \frac{P(x)}{x}$. Let M be the median of $\{f(1), f(2), \ldots, f(2024)\}$. If you submit X, you will recieve $\max(\lfloor 30 \cdot (1 - |X - M| \cdot 20) \rfloor, 0)$ points