Accuracy Round Solutions

LMT Spring 2024

May 4, 2024

1. **[6]** Compute
$$\binom{2\cdot 0\cdot 2+4!}{2^0-2+4}$$
where $\binom{m}{n}=\frac{m!}{n!(m-n)!}$.
Proposed by Aidan Duncan

Solution. $\boxed{2024}$

This evaluates out to $\binom{24}{3}=\frac{24\cdot 23\cdot 22}{6}=8\cdot 23\cdot 11=\boxed{2024}$. □

2. **[6]** Derek rolls two fair 6-sided dice. Given that he rolled at least one even number, find the probability the sum of his rolls is an even number but not a multiple of 4.
Proposed by Derek Zhao

Solution. $\boxed{\dfrac{4}{27}}$

There are 4 ways to roll a sum that is $2 \pmod 4$, which are $(2,4)$, $(6,4)$, $(4,2)$, and $(4,6)$. There are $6\cdot 6-3\cdot 3=27$ ways to roll at least 1 even number, giving the answer $\boxed{\dfrac{4}{27}}$. □

3. **[8]** Consider equilateral triangle POV and square $BENY$ centered at O, both with side length 4. Find the area of the intersection of the two polygons, given that BE is parallel to PO.
Proposed by Calvin Garces

Solution. $\boxed{4-\dfrac{2\sqrt{3}}{3}}$

The shape is a square with side length 2 with a missing $\frac{2}{\sqrt{3}}-2-\frac{4}{\sqrt{3}}$ triangle, which gives an area of $\boxed{4-\dfrac{2\sqrt{3}}{3}}$. □

4. **[8]** Find the least positive integer with strictly more 2-digit factors than 1-digit factors.
Proposed by Muztaba Syed

Solution. $\boxed{110}$

Let the number be n. Look at the factor pairs of n. For values of n less than 100 the pair in the middle (when the values are closest together) will consist of single digit numbers. For $n=100$ the middle pair is 10, 10 but we see 100 the same number of 2 digit factors as 1 digit factors.

Assuming n is not a perfect square see that our middle pair must consist of 2 distinct 2 digit numbers, and this is achieved with 10, 11. This means our answer is $\boxed{110}$. □

5. **[10]** In rectangle $ABCD$ let $AC = 4$ and $\angle ACD = 15°$. Let G be the centroid of ABC and O be the intersection of AC and BD. Let point E satisfy that $DE \parallel AC$ and lie on the circumcircle of ABC. Find the area of GEO.

   Proposed by Jiwu Jang, Corey Zhao, Derek Zhao

   Solution. $\boxed{\frac{\sqrt{3}}{3}}$

   Notice $[DEO] = 3[GEO]$ because $DO = OB = 3OG$. Additionally $\angle AOD = \angle DOC = 30°$. This means $\angle DOE = 120°$. Therefore, $[GEO] = \frac{1}{3} \cdot 2^2 \cdot \frac{1}{2} \cdot \sin(120°) = \boxed{\frac{\sqrt{3}}{3}}$.

   Remark: The three problem authors listed here were asked to write a geo problem so they each added a random point/shape to the diagram until they had a question. Each of them added one of the three points of GEO. □

6. **[10]** The unique real solution to $x^3 = 9x + 14$ can be expressed as $\sqrt[3]{m+\sqrt{n}} + \sqrt[3]{m-\sqrt{n}}$ for positive integers m and n. What is $m+n$?

   Proposed by Peter Bai

   Solution. $\boxed{29}$

   Notice that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3(a+b)ab$. We can set $x = \sqrt[3]{n+\sqrt{m}} + \sqrt[3]{n-\sqrt{m}}$ and take the cube of both sides to get $x^3 = (n+\sqrt{m}) + (n-\sqrt{m}) + 3x\sqrt[3]{n^2-m} \implies x^3 = (3\sqrt[3]{n^2-m})x + 2n$.

   We can then plug in $9 = 3\sqrt[3]{n^2-m}$ and $14 = 2n$ to get $n = 7$ and $m = 22$, which gives us our answer of $7 + 22 = \boxed{29}$. □

7. **[12]** Six people numbered 1, 2, ..., 6 are randomly arranged in a circle. Each person begins with exactly one coin. Starting with person $k = 1$ and ending with person $k = 5$, the person numbered k gives all of the coins in their possession to the person immediately clockwise. After this process is over, compute the expected number of coins in person 6's possession.

   Proposed by Muztaba Syed

   Solution. $\boxed{\frac{163}{60}}$

   Fix the location of person 6. The coin that starts immediately counter-clockwise to them will necessarily be passed it to them. The coin before that will be passed to person 6 if and only if the two people before person 6 are in order (probability $\frac{1}{2}$. Likewise the coin 3 before person 6 has probability $\frac{1}{3!}$ since the 3 people must be in order.

   By Linearity of Expectation we can add these probabilities to get

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = \boxed{\frac{163}{60}}.$$

   □

8. **[12]** Find the number of ordered pairs (m, n) of positive integers such that $2 \le m, n \le 100$ and

$$n! \equiv 30(n-1) \pmod{m!}.$$

   Proposed by Evin Liang

   Solution. $\boxed{344}$

   We split two cases. If $n \ge m$, then $n! \equiv 0 \mod m!$. Therefore, $m!$ divides $30(n-1)$. We have $30(n-1) \le 2970 < 7!$, so $m \le 6$. If m is 2 or 3, every value of n works that is at least m. If m is 4, then n must be 1 mod 4. If m is 5, then n must be 1 mod 4. If m is 6, then n must be 1 mod 24. In total, this gives $99 + 98 + 24 + 24 + 4 = 249$ pairs.

   The next case is when $m > n$. We check the cases where $m < 6$ manually and there are no solutions. If $m \ge 6$, then $n! < m!$ and $30(n-1) < m!$, so $n! = 30(n-1)$. So the solutions are $n = 5$ and $m \ge 6$. This gives 95 solutions, for a total of $249 + 95 = \boxed{344}$ solutions. □

9. **[14]** Consider isosceles triangle ABC with $\angle A = 72^\circ$ and $AB = AC$. Let D be a point on (ABC) and denote by I_1 and I_2 the incenters of ABD and ACD, respectively. Let points E and F be the intersections of rays $\overrightarrow{DI_1}$ and $\overrightarrow{DI_2}$ with minor arcs AB and AC, respectively. Given that $\triangle ABC$ has inradius 5, find the length of the locus of the incenter of $\triangle DEF$ as D varies along minor arc BC. Express your answer in simplest form.

   Proposed by Jerry Xu

   Solution. $\boxed{4\pi\cos(27^\circ)\csc(72^\circ)}$

   Denote the incenter of ABC as I and the incenter of DEF as I_3. We propose the following claim:

   Claim. The desired locus is the arc of the circle centered at A with radius $AE = AF$ that subtends an angle of $\angle A = 72^\circ$.

   Proof. Since DI_1 is the angle bisector of $\angle ADB$ and DI_2 is the angle bisector of $\angle ADC$, E and F are the midpoints of minor arc AB and AC. As ABC is isosceles, $AE = AF$. Thus, AD is the angle bisector of $\angle EDF$. By Fact 5, we must have $AE = AI_3 = AF$: also, since D is varied along minor arc BC, I_3 must lie in the interior of ABC. This is sufficient to prove the claim. ■

   Finding AE clearly finishes the problem. We now propose the following:

   Claim. In triangle ABC with incenter I, denote by M_A, M_B, M_C the midpoints of arcs BC, CA, AB, respectively. Then M_AM_B is the perpendicular bisector of CI and analogously for M_BM_C and M_CM_A.

   Proof. Consider M_A and M_B: by Fact 5 where M_A is the center of (BIC), we have that $M_BC = M_BI$: similarly, by Fact 5 where M_B is the center of (A_C), we have that $M_AC = M_AI$. Since $M_AM_B = M_AM_B$, we have $M_ACM_B \cong M_AIM_B$ by SSS congruence. The result follows. ■

   Now we compute AE. Note that $BC = 5\cot(27^\circ)\cdot 2$, and by Law of Sines we have

$$\frac{BC}{\sin(72^\circ)} = \frac{AE}{\sin(27)}.$$

   From here we get $AE = \frac{10\cot(27^\circ)\sin(27^\circ)}{\sin(72^\circ)} = \frac{10\cos(27^\circ)}{\sin(72^\circ)}$. This means our answer is

$$10\cos(27^\circ)\csc(72^\circ)\cdot\frac{2\pi}{5} = \boxed{4\pi\cos(27^\circ)\csc(72^\circ)}$$

   Corollary. It follows by Fact 5 that $EI_1 = EA = AI_3 = AF = FI_2$. □

10. **[14]** Derek takes a random walk in the xy-plane. He starts at the origin and when he takes a step, he moves 1 unit in each of the four directions with equal probability. Let (m,n) be the point that Derek reaches after 100 steps. Compute the expected value of $(mn)^2$.

    Proposed by Evin Liang

    Solution. $\boxed{2475}$

    Let Derek's position after n steps be (x_n, y_n). Then we have

$$E(x_{n+1}^2y_{n+1}^2) = \frac{1}{4}E\Big((x_n+1)^2y_n^2 + (x_n-1)^2y_n^2 + x_n^2(y_n+1)^2 + x_n^2(y_n-1)^2\Big) = E(x_n^2y_n^2) + \frac{1}{2}E(x_n^2) + \frac{1}{2}E(y_n^2).$$

    We also have

$$E(x_{n+1}^2) = \frac{1}{4}E\Big(x_n^2 + x_n^2 + (x_n+1)^2 + (x_n-1)^2\Big) = E(x_n^2) + \frac{1}{2}.$$

    Therefore, $E(x_n^2) = E(y_n^2) = \frac{n}{2}$, so $E(x_n^2y_n^2) = \frac{n(n-1)}{4}$. Thus the answer is $\frac{100\cdot 99}{4} = \boxed{2475}$. □

11. **[TIEBREAKER]** Let $P(n)$ be the number of ways to write the positive integer n as a sum of not-necessarily distinct positive integers, where order doesn't matter. For example, $P(3) = 3$ because 3, $1+2$, and $1+1+1$ are all ways of writing 3 as a sum of positive integers. Estimate $\frac{P(2024)}{P(2023)}$ to 5 digits after the decimal point.

    Proposed by Aidan Duncan

    Solution. $\boxed{1.02842}$

    Wolfram Alpha □