Accuracy Round

LMT Spring 2024

May 4, 2024

1. **[6]** Compute
$$\binom{2\cdot 0\cdot 2+4!}{2^0-2+4}$$
where $\binom{m}{n}=\frac{m!}{n!(m-n)!}$.

2. **[6]** Derek rolls two fair 6-sided dice. Given that he rolled at least one even number, find the probability the sum of his rolls is an even number but not a multiple of 4.

3. **[8]** Consider equilateral triangle POV and square $BENY$ centered at O, both with side length 4. Find the area of the intersection of the two polygons, given that BE is parallel to PO.

4. **[8]** Find the least positive integer with strictly more 2-digit factors than 1-digit factors.

5. **[10]** In rectangle $ABCD$ let $AC=4$ and $\angle ACD=15^\circ$. Let G be the centroid of ABC and O be the intersection of AC and BD. Let point E satisfy that $DE \parallel AC$ and lie on the circumcircle of ABC. Find the area of GEO.

6. **[10]** The unique real solution to $x^3=9x+14$ can be expressed as $\sqrt[3]{m+\sqrt{n}}+\sqrt[3]{m-\sqrt{n}}$ for positive integers m and n. What is $m+n$?

7. **[12]** Six people numbered 1, 2, ..., 6 are randomly arranged in a circle. Each person begins with exactly one coin. Starting with person $k=1$ and ending with person $k=5$, the person numbered k gives all of the coins in their possession to the person immediately clockwise. After this process is over, compute the expected number of coins in person 6's possession.

8. **[12]** Find the number of ordered pairs (m,n) of positive integers such that $2\le m,n\le 100$ and
$$n!\equiv 30(n-1) \pmod{m!}.$$

9. **[14]** Consider isosceles triangle ABC with $\angle A=72^\circ$ and $AB=AC$. Let D be a point on (ABC) and denote by I_1 and I_2 the incenters of ABD and ACD, respectively. Let points E and F be the intersections of rays $\overrightarrow{DI_1}$ and $\overrightarrow{DI_2}$ with minor arcs AB and AC, respectively. Given that $\triangle ABC$ has inradius 5, find the length of the locus of the incenter of $\triangle DEF$ as D varies along minor arc BC. Express your answer in simplest form.

10. **[14]** Derek takes a random walk in the xy-plane. He starts at the origin and when he takes a step, he moves 1 unit in each of the four directions with equal probability. Let (m,n) be the point that Derek reaches after 100 steps. Compute the expected value of $(mn)^2$.

11. **[TIEBREAKER]** Let $P(n)$ be the number of ways to write the positive integer n as a sum of not-necessarily distinct positive integers, where order doesn't matter. For example, $P(3)=3$ because 3, $1+2$, and $1+1+1$ are all ways of writing 3 as a sum of positive integers. Estimate $\frac{P(2024)}{P(2023)}$ to 5 digits after the decimal point.