Team Round Solutions

LMT "Fall"

December 17, 2022

1. **[9]** Let x be the positive integer satisfying $5^2 + 28^2 + 39^2 = 24^2 + 35^2 + x^2$. Find x.

   Proposed by Raymond Xu

   Solution. $\boxed{23}$

   Doing arithmetic with or without difference of squares gives $5^2 + 28^2 + 39^2 - 24^2 - 35^2 = 529$, so the answer is $\boxed{23}$. □

2. **[10]** Ada rolls a standard 4-sided die 5 times. The probability that the die lands on at most two distinct sides can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

   Proposed by Tianyi Zhou

   Solution. $\boxed{23128}$

   Total possibilities: $4^5 = 1024$ Good possibilities: 6 choices for the buildings and 2^5 ways for the classes to be in those buildings. However, this counts each one-building possibility 3 times, which is 2 too many, so there are $6 * 2^5 - 8 = 184$ ways for her classes to be in 2 or fewer buildings. The answer is $\frac{184}{1024} = \frac{23}{128}$, giving an answer of $\boxed{23128}$ □

3. **[11]** Billiam is distributing his ample supply of balls among an ample supply of boxes. He distributes the balls as follows: he places a ball in the first empty box, and then for the greatest positive integer n such that all n boxes from box 1 to box n have at least one ball, he takes all of the balls in those n boxes and puts them into box $n+1$. He then repeats this process indefinitely. Find the number of repetitions of this process it takes for one box to have at least 2022 balls.

   Proposed by Jerry Xu

   Solution. $\boxed{2048}$

   Note that the ball in the first box is always moved to a different box by the end of any given process, so Billiam is always placing a ball into the first box. Now, we can bash out that his first 8 moves are like so:

   $$\begin{aligned}
   &\text{turn 1: } 1 \longrightarrow 0,1\\
   &\text{turn 2: } 1,1 \longrightarrow 0,0,2\\
   &\text{turn 3: } 1,0,2 \longrightarrow 0,1,2\\
   &\text{turn 4: } 1,1,2 \longrightarrow 0,0,0,4\\
   &\text{turn 5: } 1,0,0,4 \longrightarrow 0,1,0,4\\
   &\text{turn 6: } 1,1,0,4 \longrightarrow 0,0,2,4\\
   &\text{turn 7: } 1,0,2,4 \longrightarrow 0,1,2,4\\
   &\text{turn 8: } 1,1,2,4 \longrightarrow 0,0,0,0,8
   \end{aligned}$$

   We can notice that on turn 2^n we first get a box with 2^n balls, with $n+1$ zeroes leading that box. We can note why this happens: if we have a setup like so:

   $$\text{turn } k\text{: } 0,0,0,\ldots,0,2^n$$

Then we'll have

$$\text{turn } k+1\text{: } 1,0,0,\ldots,0,2^n \longrightarrow 0,1,0\ldots,0,2^n$$
$$\text{turn } k+2\text{: } 1,1,0,\ldots,0,2^n \longrightarrow 0,0,2\ldots,0,2^n$$

This pattern continues in the same way that it did in the first 8 moves: after turn k, we get one 1, then two that form a 2 in the third place, and then we get two 1s and the 2 forming a 4 in the fourth place, and then two 1s and a 2 and the 4 forming an 8 and so on. We can thus therefore say that the following statements holds:

$$\text{turn 2048: } 0,0,\ldots,0,2048$$

Note that on every turn before 2048, the maximum number of balls in any box was 1024. Therefore, our answer is $\boxed{2048}$. □

4. **[12]** Find the least positive integer ending in 7 with exactly 12 positive divisors.

Proposed by William Hua

Solution. $\boxed{1197}$

The answer must be in the form x^2yz, x^3y^2, x^5y, or x^{11}, such that x, y, and z are distinct prime numbers. The number must not be divisible by 2 or 5 because the number ends in 7.

Let N be the answer. We split this into three cases:

Case 1: the answer is in the form x^2yz. We want the number to be 7 mod 10. If we let $x = 3$, then $N = 9yz$, with $y,z \geq 7$. Without loss of generality, assume $y < z$. Aware that the number must end in 7, if $y = 7$, we get $N = 9\cdot 7\cdot 19$ as the smallest value ($9\cdot 7 \equiv 3 \mod 10$, and the one number z mod 10 that yields $3x \equiv 7$ is 9, so $z \equiv 9$). If $y = 11$, then $N = 9\cdot 11\cdot 13$. Note that if $y \geq 13$, then $N > 9\cdot 11\cdot 13$, since $z > y$. Therefore, if $x = 3$, then $N = \min(9\cdot 7\cdot 19, 9\cdot 11\cdot 13) = 1197$. If $x = 7$, then $N = 49yz$, such that $y,z \neq 7$. Note that the smallest possible N in this form is $49\cdot 3\cdot 11$, which is already larger than 1197. Attempting any value of x will guarantee $N \geq 121\cdot 3\cdot 7 = 2541$, so the smallest N in the form x^2yz is 1197.

Case 2: the answer is in the form x^3y^2. If $x = 3$, then $x^3y^2 = 27y^2$, so $y^2 \equiv 1 \mod 10$, so $y \equiv 1,9 \mod 10$. Therefore, $y = 11$ to yield the smallest $27y^2$, resulting in $27\cdot 121 = 3267$, which is too large. In fact, any x^3y^2 that isn't divisible by 2 or 5 will be at least $3^3 * 7^2 = 1323$, so we yield a smaller N from case 1.

Case 3: the answer is in the form x^5y. Then, because N isn't divisible by 2 or 5, x^5y is at least $3^5\cdot 7 = 1701$, which is bigger than 1197. So N isn't in the form x^5y.

Case 4: the answer is in the form x^{11}. This is too large. In fact, if we even let $x = 2$, the smallest prime number, then $2^{11} > 1197$.

Therefore, the answer is $\boxed{1197}$. □

5. **[13]** Let H be a regular hexagon with side length 1. The sum of the areas of all triangles whose vertices are all vertices of H can be expressed as $A\sqrt{B}$ for positive integers A and B such that B is square-free. What is $1000A+B$?

Proposed by Brandon Ni

Solution. $\boxed{9003}$

There are $\binom{6}{3} = 20$ triangles that can be made by the vertices of the regular hexagon. 3 distinct triangles that can be formed using the vertices: – 6 triangles with side lengths $1,1,\sqrt{3}$ and area $\frac{\sqrt{3}}{4}$ – 12 triangles with side lengths $1,\sqrt{3},2$ and area $\frac{\sqrt{3}}{2}$ – 2 triangles with side lengths $\sqrt{3},\sqrt{3},\sqrt{3}$ and area $\frac{3\sqrt{3}}{4}$

The sum of all the areas is $6\cdot\frac{\sqrt{3}}{4}+12\cdot\frac{\sqrt{3}}{2}+2\cdot\frac{3\sqrt{3}}{4} = 9\sqrt{3}$ $1000a+b = \boxed{9003}$. □

6. **[15]** An isosceles trapezoid $PQRS$, with $\overline{PQ} = \overline{QR} = \overline{RS}$ and $\angle PQR = 120°$, is inscribed in the graph of $y = x^2$ such that QR is parallel to the x-axis and R is in the first quadrant. The x-coordinate of point R can be written as $\frac{\sqrt{A}}{B}$ for positive integers A and B such that A is square-free. Find $1000A+B$.

Proposed by Hannah Shen

Solution. $\boxed{3003}$

Since the trapezoid is isosceles, $\angle QRS = 120^\circ$ as well. R has coordinates (X, X^2). The x-coordinate of point S must be twice that of R, so S must be at $(2X, 4X^2)$. The difference in their y-coordinates must be $\sqrt{3} * X$; in other words, $3X^2 = \sqrt{3} * X$. Solving, $X = \frac{\sqrt{3}}{3}$, so $A + B = \boxed{6}$. □

7. **[17]** A regular hexagon is split into 6 congruent equilateral triangles by drawing in the 3 main diagonals. Each triangle is colored 1 of 4 distinct colors. Rotations and reflections of the figure are considered nondistinct. Find the number of possible distinct colorings.

Proposed by Jerry Xu

Solution. $\boxed{430}$

There are six rotations of $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ$, and 300° of the hexagon. There are also six reflections of the hexagon. Therefore, we divide our final answer by 12. Now we shall split into the cases.

Case 1. Identity Since there are no restrictions, there are $4^6 = 4096$ colorings for this case.

Case 2. Rotation by 60° Note that each triangle maps to an adjacent triangle so therefore the first triangle fixes the rest of them, so there are 4 colorings here.

Case 3. Rotation by 120° There are two cycles of three triangles which fix each other, so there are $4^2 = 16$ colorings here.

Case 4. Rotation by 180° There are three pairs of triangles which fix each other, so there are $4^3 = 64$ colorings here.

Case 5. Rotation by 240° This is equivalent to rotating 120° in the reverse direction so there exists a bijection in the number of colorings. Therefore, our total colorings for this case is 16.

Case 6. Rotation by 300° This case also has a bijection with rotation by 60° so the total colorings for this case is 4.

Case 7. Reflection along line between opposite vertices Here there are three pairs of triangles that fix each other so there are $4^3 = 64$ colorings here. There are three such lines so our total is $3 \cdot 64 = 192$.

Case 8. Reflection along line between midpoints of opposite sides Here two pairs of triangles fix each other and and the other two can have any colorings, so our colorings for this case is $4^4 = 256$. There are three such lines so our total is $256 \cdot 3 = 768$.

Thus our final answer is $\frac{4096+4+16+64+16+4+192+768}{12} = \boxed{430}$ □

8. **[19]** An odd positive integer n can be expressed as the sum of two or more consecutive integers in exactly 2023 ways. Find the greatest possible nonnegative integer k such that 3^k is a factor of the least possible value of n.

Proposed by Jerry Xu

Solution. $\boxed{22}$

Note that if the sum of the integers $x+1, x+2, ..., x+m$ is n, then their average is $\frac{n}{m}$. This average is either an integer of has a fractional part of $\frac{1}{2}$. Since n is odd, the average is only an integer if m is odd. The average has a fractional part of $\frac{1}{2}$ if m is even and m is exactly double an odd divisor of n (thus resulting in an odd divded by 2, which leaves a fractional part of $\frac{1}{2}$). We can therefore note that there exists a bijection between the number of cases where the average is an integer and the number of cases where the average has a fractional part of $\frac{1}{2}$. Therefore, the total number of ways to express n as a sum of consecutive integers is twice the number of odd factors of n. Since the question requires n to be the sum of two or more consecutive integers, we must subtract off 1 from this count to account for the case where n is expressed as simply n. Therefore, we have that n has 2024 factors. 2024 is factorized as $2^3 \cdot 11 \cdot 23$. We can count odd factors of a number $m = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$ for odd p_i as $(e_1+1)(e_2+1)\dots(e_n+1) = 2^3 \cdot 11 \cdot 23$. Therefore, one of the terms on the LHS must be one less than a multiple of 23 and since n is maximized when the largest e are paired to the smallest p, we have that our answer is $23 - 1 = \boxed{22}$. □

9. **[21]** In isosceles trapezoid $ABCD$ with $AB < CD$ and $BC = AD$, the angle bisectors of $\angle A$ and $\angle B$ intersect CD at E and F respectively, and intersect each other outside the trapezoid at G. Given that $AD = 8$, $EF = 3$, and $EG = 4$, the

area of $ABCD$ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c, with a and c relatively prime and b squarefree. Find $10000a+100b+c$.

Proposed by Muztaba Syed

Solution. $\boxed{1595516}$

Let $\angle BAE = a$. $\angle ADC = 180-2a$ so $\angle AED = a$. This means that $AD = AE = BC = CF$. We also see that $\angle FEG = \angle EFG = a$, so $\triangle ADE \sim \triangle FGE \sim \triangle FCB \sim \triangle BGA$. We have $FG = 4$, so $DE = FC = 8$ and $AE = AF = 6$. We see ABG is $\frac{10}{4} = \frac{5}{2}$ times EFG so the area is $\frac{25}{4}$ times the area of EFG. ADE and BFC have 4 times the area of EFG. The desired value is: $[ADE]+[BFC]+[ABG]-[EFG] = \frac{53}{4}[EFG]$. We see $[EFG] = \frac{3\sqrt{55}}{4}$, so our answer is $\frac{159\sqrt{55}}{16} \implies \boxed{1595516}$. □

10. **[23]** Let $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\cos(\alpha n+\beta m)}{2^n 3^m}.$$

can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Evin Liang

Solution. $\boxed{15013}$

Consider the sum $S = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{i(\alpha n+\beta m)}}{2^n 3^m}$. Because $e^{i(\alpha n+\beta m)} = \cos(\alpha n+\beta m)+i\sin(\alpha n+\beta m)$, the given sum is the real part of S. Rewriting the summand as $\left(\frac{e^{i\alpha}}{2}\right)^n\left(\frac{e^{i\beta}}{3}\right)^m$, we can factor the sum $S = \sum_{n=0}^{\infty}\left(\frac{e^{i\alpha}}{2}\right)^n \sum_{m=0}^{\infty}\left(\frac{e^{i\beta}}{3}\right)^m$. This is a product of two geometric series $\frac{1}{1-\frac{e^{i\alpha}}{2}}\cdot\frac{1}{1-\frac{e^{i\beta}}{3}}$. Multiplying the numerator and denominator by $36\left(1-\frac{e^{-i\alpha}}{2}\right)\left(1-\frac{e^{-i\beta}}{3}\right)$ we have $S = \frac{4-2e^{-i\alpha}}{4-2e^{i\alpha}-2e^{-i\alpha}+1}\cdot\frac{9-3e^{-i\beta}}{9-3e^{i\beta}-3e^{-i\beta}+1}$. Now $e^{i\alpha}+e^{-i\alpha} = 2\cos(\alpha) = \frac{6}{5}$ and $e^{i\beta}+e^{-i\beta} = 2\cos(\beta) = \frac{8}{5}$. This makes $S = \frac{(4-2e^{-i\alpha})(9-3e^{-i\beta})}{338/25}$. The numerator expands as $36-18e^{-i\alpha}-12e^{-i\beta}+6e^{i(\alpha+\beta)}$. The real part is $36-18\cos(\alpha)-12\cos(\beta)+6\cos(\alpha+\beta) = \frac{78}{5}$ (remembering $\alpha+\beta = 90°$) and so the sum is $\frac{15}{13}$, which produces an answer of $\boxed{15013}$ □