Guts Round

Lexington High School

December 12th, 2020

10th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** Find the remainder when 2020! is divided by 2020^2.

______________ 2. **[5]** In a five term arithmetic sequence, the first term is 2020 and the last term is 4040. Find the second term of the sequence.

______________ 3. **[5]** Circles C_1, C_2, and C_3 have radii 2, 3, and 6, and respectively. If the area of a fourth circle C_4 is the sum of the areas of C_1, C_2, and C_3, compute the radius of C_4.

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Team Name: ______________________________

______________ 4. **[5]** At Lexington High School, each student is given a unique five-character ID consisting of uppercase letters. Compute the number of possible IDs that contain the string "LMT".

______________ 5. **[5]** For what digit d is the base 9 numeral $7d35_9$ divisible by 8?

______________ 6. **[5]** The number 2021 can be written as the sum of 2021 consecutive integers. What is the largest term in the sequence of 2021 consecutive integers?

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______________ 7. **[6]** $2020 \cdot N$ is a perfect integer cube. If N can be expressed as $2^a \cdot 5^b \cdot 101^c$, find the least possible value of $a+b+c$ such that a, b, c, are all positive integers and not necessarily distinct.

______________ 8. **[6]** A rhombus with side length 1 has an inscribed circle with radius $\frac{1}{3}$. If the area of the rhombus can be expressed as $\frac{a}{b}$ for relatively prime, positive a, b, evaluate $a+b$.

______________ 9. **[6]** If $xy : yz : zx = 6 : 8 : 12$, and $x^3 + y^3 + z^3 : xyz$ is $m : n$ where m and n are relatively prime positive integers, then find $m+n$.

10th Annual Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[6]** 2020 magicians are divided into groups of 2 for the Lexington Magic Tournament. After every 5 days, which is the duration of one match, teams are rearranged so no 2 people are ever on the same team. If the longest tournament is n days long, what is the value of n?

______________ 11. **[6]** Cai and Lai are eating cookies. Their cookies are in the shape of two regular hexagons glued to each other, and the cookies have area 18 square units. They each make a cut along the two long diagonals of a cookie; this now makes four pieces for them to eat and enjoy. What is the minimum area among the four pieces?

______________ 12. **[6]** If the value of the infinite sum

$$\frac{1}{2^2-1^2}+\frac{1}{4^2-2^2}+\frac{1}{8^2-4^2}+\frac{1}{16^2-8^2}+\cdots.$$

can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b, evaluate $a+b$.

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Team Name: ______________________________

______________ 13. **[7]** Let set S contain all positive integers that are one less than a perfect square. Find the sum of all powers of 2 that can be expressed as the product of two (not necessarily distinct) members of S.

______________ 14. **[7]** Ada and Emily are playing a game that ends when either player wins, after some number of rounds. Each round, either nobody wins, Ada wins, or Emily wins. The probability that neither player wins each round is $\frac{1}{5}$ and the probability that Emily wins the game as a whole is $\frac{3}{4}$. If the probability that in a given round Emily wins is $\frac{m}{n}$ such that m and n are relatively prime integers, then find $m+n$.

______________ 15. **[7]** $\triangle ABC$ has $AB=5$, $BC=6$, and $AC=7$. Let M be the midpoint of BC, and let the circumcircle of $\triangle ABM$ intersect AC at N. If the length of segment MN can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b, find $a+b$.

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Team Name: ______________________________

______________ 16. **[7]** Compute

$$\frac{2019!\cdot 2^{2019}}{(2020^2-2018^2)(2020^2-2016^2)\cdots(2020^2-2^2)}.$$

______________ 17. **[7]** In a regular square room of side length $2\sqrt{2}$ ft, two cats that can see 2 feet ahead of them are randomly placed into the four corners such that they do not share the same corner. If the probability that they don't see the mouse, also placed randomly into the room, can be expressed as $\frac{a-b\pi}{c}$, where a, b, c are positive integers with a greatest common factor of 1, then find $a+b+c$.

______________ 18. **[7]** Given that $\sqrt{x+2y}-\sqrt{x-2y}=2$, compute the minimum value of $x+y$.

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Team Name: ______________________________

______________ 19. **[8]** Find the second smallest prime factor of $18! + 1$.

______________ 20. **[8]** Cyclic quadrilateral $ABCD$ has $AC = AD = 5, CD = 6$, and $AB = BC$. If the length of AB can be expressed as $\frac{a\sqrt{b}}{c}$ where a, c are relatively prime positive integers and b is square-free, evaluate $a + b + c$.

______________ 21. **[8]** A sequence with first term a_0 is defined such that $a_{n+1} = 2a_n^2 - 1$ for $n \geq 0$. Let N denote the number of possible values of a_0 such that $a_0 = a_{2020}$. Find the number of factors of N.

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______________ 22. **[8]** Find the area of a triangle with side lengths $\sqrt{13}$, $\sqrt{29}$, and $\sqrt{34}$. The area can be expressed as $\frac{m}{n}$ for m, n relatively prime positive integers, then find $m + n$.

______________ 23. **[8]** Let $f : \mathbb{R}\backslash 0 \to \mathbb{R}\backslash 0$ be a non-constant, continuous function defined such that $f(3^x 2^y) = \frac{y}{x} f(2^x) + \frac{x}{y} f(3^y)$ for any $x, y \neq 0$. Compute $\frac{f(1296)}{f(6)}$.

______________ 24. **[8]** In the Oxtingnle math team, there are 5 students, numbered 1 to 5, all of which either always tell the truth or always lie. When Marpeh asks the team about how they did in a 10 question competition, each student i makes 5 separate statements (so either they are all false or all true): "I got problems $i + 1$ to $2i$, inclusive, wrong", and then "Student j got both problems i and $2i$ correct" for all $j \neq i$. What is the most problems the team could have gotten correctly?

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Team Name: ______________________________

______________ 25. **[9]** Consider the equation $x^4 - 24x^3 + 210x^2 + mx + n = 0$. Given that the roots of this equation are nonnegative reals, find the maximum possible value of a root of this equation across all values of m and n.

______________ 26. **[9]** Let ω_1 and ω_2 be two circles with centers O_1 and O_2. The two circles intersect at A and B. ℓ is the circles' common external tangent that is closer to B, and it meets ω_1 at T_1 and ω_2 at T_2. Let C be the point on line AB not equal to A that is the same distance from ℓ as A is. Given that $O_1O_2 = 15$, $AT_1 = 5$ and $AT_2 = 12$, find $AC^2 + T_1T_2{}^2$.

______________ 27. **[9]** A list consists of all positive integers from 1 to 2020, inclusive, with each integer appearing exactly once. Define a move as the process of choosing four numbers from the current list and replacing them with the numbers $1, 2, 3, 4$. If the expected number of moves before the list contains exactly two 4's can be expressed as $\frac{a}{b}$ for relatively prime positive integers, evaluate $a + b$.

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______________ 28. **[11]** 13 LHS Students attend the LHS Math Team tryouts. The students are numbered $1,2,..13$. Their scores are $s_1, s_2, ...s_{13}$, respectively. There are 5 problems on the tryout, each of which is given a weight, labelled $w_1, w_2, ...w_5$. Each score s_i is equal to the sums of the weights of all problems solved by student i. On the other hand, each weight w_j is assigned to be $\frac{1}{\sum s_i}$, where the sum is over all the scores of students who solved problem j. (If nobody solved a problem, the score doesn't matter). If the largest possible average score of the students can be expressed in the form $\frac{\sqrt{a}}{b}$, where a is square-free, find $a+b$.

______________ 29. **[11]** Find the number of pairs of integers (a,b) with $0 \le a, b \le 2019$ where $ax \equiv b \pmod{2020}$ has exactly 2 integer solutions $0 \le x \le 2019$.

______________ 30. **[11]** $\triangle ABC$ has the property that $\angle ACB = 90°$. Let D and E be points on AB such that D is on ray BA, E is on segment AB, and $\angle DCA = \angle ACE$. Let the circumcircle of $\triangle CDE$ hit BC at $F \ne C$, and EF hit AC and DC at P and Q, respectively. If $EP = FQ$, then the ratio $\frac{EF}{PQ}$ can be written as $a+\sqrt{b}$ where a and b are positive integers. Find $a+b$.

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Team Name: ______________________________

______________ 31. **[13]** Let real angles $\theta_1, \theta_2, \theta_3, \theta_4$ satisfy

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \sin\theta_4 = 0,$$
$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos\theta_4 = 0.$$

If the maximum possible value of the sum

$$\sum_{i<j} \sqrt{1 - \sin\theta_i \sin\theta_j - \cos\theta_i \cos\theta_j}$$

for $i, j \in \{1,2,3,4\}$ can be expressed as $a + b\sqrt{c}$, where c is square-free and a, b, c are positive integers, find $a+b+c$

______________ 32. **[13]** In a lottery there are 14 balls, numbered from 1 to 14. Four of these balls are drawn at random. D'Angelo wins the lottery if he can split the four balls into two disjoint pairs, where the two balls in each pair have difference at least 5. The probability that D'Angelo wins the lottery can be expressed as $\frac{m}{n}$, with m, n relatively prime. Find $m+n$.

______________ 33. **[13]** Let ω_1 and ω_2 be two circles that intersect at two points: A and B. Let C and E be on ω_1, and D and F be on ω_2 such that CD and EF meet at B and the three lines CE, DF, and AB concur at a point P that is closer to B than A. Let Ω denote the circumcircle of $\triangle DEF$. Now, let the line through A perpendicular to AB hit EB at G, GD hit Ω at J, and DA hit Ω again at I. A point Q on IE satisfies that $CQ = JQ$. If $QJ = 36$, $EI = 21$, and $CI = 16$, then the radius of Ω can be written as $\frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, b is not divisible by the square of a prime, and $\gcd(a,c) = 1$. Find $a+b+c$.

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Team Name: ______________________________

______________ 34. **[15]** Your answer to this problem will be an integer between 0 and 100, inclusive. From all the teams who submitted an answer to this problem, let the average answer be A. Estimate the value of $\left\lfloor \frac{2}{3}A \right\rfloor$. If your estimate is E and the answer is A, your score for this problem will be

$$\max(0, \lfloor 15 - 2 \cdot |A - E| \rfloor).$$

______________ 35. **[15]** Estimate the number of ordered pairs (p, q) of positive integers at most 2020 such that the cubic equation $x^3 - px - q = 0$ has three distinct real roots. If your estimate is E and the answer is A, your score for this problem will be

$$\left\lfloor 15 \min\left(\frac{A}{E}, \frac{E}{A}\right)\right\rfloor.$$

______________ 36. **[15]** Estimate the product of all the nonzero digits in the decimal expansion of 2020!. If your estimate is E and the answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 0.02 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$