LMT Fall Online: Division A

December 5th – December 12th, 2020

Contest Instructions

Contest Window

The Team Round consists of 30 short answer problems, including 10 theme problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Saturday, December 5th, at 3:00 pm**. Teams will have until **Saturday, December 12th, at 3:00 pm** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the sum of the point values of the problems you answered correctly. Note that the theme problems deviate from the general trend in point values. Ties will not be broken. Results will be posted shortly after the competition, and the top teams will be recognized.

1. **[10]** Ben writes the string
$$\underbrace{111\ldots11}_{2020\text{ digits}}$$
on a blank piece of paper. Next, in between every two consecutive digits, he inserts either a plus sign (+) or a multiplication sign (×). He then computes the expression using standard order of operations. Find the number of possible distinct values that Ben could have as a result.

2. **[10]** 1001 marbles are drawn at random and without replacement from a jar of 2020 red marbles and n blue marbles. Find the smallest positive integer n such that the probability that there are more blue marbles chosen than red marbles is strictly greater than $\frac{1}{2}$.

3. **[10]** Find the value of $\lfloor\frac{1}{6}\rfloor+\lfloor\frac{4}{6}\rfloor+\lfloor\frac{9}{6}\rfloor+\cdots+\lfloor\frac{1296}{6}\rfloor$.

4. **[10]** Let $\triangle ABC$ with $AB=AC$ and $BC=14$ be inscribed in a circle ω. Let D be the point on ray BC such that $CD=6$. Let the intersection of AD and ω be E. Given that $AE=7$, find AC^2.

5. **[15]** Ada is taking a math test from 12:00 to 1:30, but her brother, Samuel, will be disruptive for two ten-minute periods during the test. If the probability that her brother is not disruptive while she is solving the challenge problem from 12:45 to 1:00 can be expressed as $\frac{m}{n}$, find $m+n$.

6. **[15]** Circle ω has radius 10 with center O. Let P be a point such that $PO=6$. Let the midpoints of all chords of ω through P bound a region of area R. Find the value of $\lfloor 10R\rfloor$.

7. **[15]** Let S denote the sum of all rational numbers of the form $\frac{a}{b}$, where a and b are relatively prime positive divisors of 1300. If S can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, then find $m+n$.

8. **[15]** Find the sum of all positive integers a such that there exists an integer n that satisfies the equation:
$$a!\cdot 2^{\lfloor\sqrt{a}\rfloor}=n!.$$

9. **[20]** $\triangle ABC$ has a right angle at B, $AB=12$, and $BC=16$. Let M be the midpoint of AC. Let ω_1 be the incircle of $\triangle ABM$ and ω_2 be the incircle of $\triangle BCM$. The line externally tangent to ω_1 and ω_2 that is not AC intersects AB and BC at X and Y, respectively. If the area of $\triangle BXY$ can be expressed as $\frac{m}{n}$, compute is $m+n$.

10. **[20]** Define a sequence $\{a_n\}_{n\geq 1}$ recursively by $a_1=1$, $a_2=2$, and for all integers $n\geq 2$, $a_{n+1}=(n+1)^{a_n}$. Determine the number of integers k between 2 and 2020, inclusive, such that $k+1$ divides a_k-1.

11. **[20]** Two sequences of nonzero reals $a_1,a_2,a_3,\ldots$ and $b_2,b_3,\ldots$ are such that $b_n=\prod_{i=1}^{n}a_i$ and $a_n=\frac{b_n^2}{3b_n-3}$ for all integers $n>1$. Given that $a_1=\frac{1}{2}$, find $|b_{60}|$.

12. **[20]** Richard comes across an infinite row of magic hats, $H_1,H_2,\ldots$ each of which may contain a dollar bill with probabilities $p_1,p_2,\ldots$. If Richard draws a dollar bill from H_i, then $p_{i+1}=p_i$, and if not, $p_{i+1}=\frac{1}{2}p_i$. If $p_1=\frac{1}{2}$ and E is the expected amount of money Richard makes from all the hats, compute $\lfloor 100E\rfloor$.

13. **[25]** Find the number of integers n from 1 to 2020 inclusive such that there exists a multiple of n that consists of only 5's.

14. **[25]** Two points E and F are randomly chosen in the interior of unit square $ABCD$. Let the line through E parallel to AB hit AD at E_1, the line through E parallel to AD hit CD at E_2, the line through F parallel to AB hit BC at F_1, and the line through F parallel to BC hit AB at F_2. The expected value of the overlap of the areas of rectangles EE_1DE_2 and FF_1BF_2 can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

15. **[25]** Let x satisfy $x^4+x^3+x^2+x+1=0$. Compute the value of $(5x+x^2)(5x^2+x^4)(5x^3+x^6)(5x^4+x^8)$.

16. **[25]** Two circles ω_1 and ω_2 have centers O_1 and O_2, respectively, and intersect at points M and N. The radii of ω_1 and ω_2 are 12 and 15, respectively, and $O_1O_2=18$. A point X is chosen on segment MN. Line O_1X intersects ω_2 at points A and C, where A is inside ω_1. Similarly, line O_2X intersects ω_1 at points B and D, where B is inside ω_2. The perpendicular bisectors of segments AB and CD intersect at point P. Given that $PO_1=30$, find PO_2^2.

17. **[30]** There are n ordered tuples of positive integers (a,b,c,d) that satisfy
$$a^2+b^2+c^2+d^2=13\cdot 2^{13}.$$
Let these ordered tuples be $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2),\dots,(a_n,b_n,c_n,d_n)$. Compute $\sum_{i=1}^{n}(a_i+b_i+c_i+d_i)$.

18. **[30]** Let there be a polynomial f of degree at most 13 such that $f(k)=13^k$ for $0\le k\le 13$. Compute the last three digits of $f(14)$.

19. **[30]** Euhan and Minjune are playing a game. They choose a number N so that they can only say integers up to N. Euhan starts by saying the 1, and each player takes turns saying either $n+1$ or $4n$ (if possible), where n is the last number said. The player who says N wins. What is the smallest number larger than 2019 for which Minjune has a winning strategy?

20. **[30]** Let $ABCD$ be a cyclic quadrilateral with center O with $AB>CD$ and $BC>AD$. Let M and N be the midpoint of sides AD and BC, respectively, and let X and Y be on AB and CD, respectively, such that $AX\cdot CY=BX\cdot DY=20000$, and $AX\le CY$. Let lines AD and BC hit at P, and let lines AB and CD hit at Q. The circumcircles of $\triangle MNP$ and $\triangle XYQ$ hit at a point R that is on the opposite side of CD as O. Let R_1 be the midpoint of PQ and B, D, and R be collinear. Let O_1 be the circumcenter of $\triangle BPQ$. Let the lines BO_1 and DR_1 intersect at a point I. If $BP\cdot BQ=823875$, $AB=429$, and $BC=495$, then $IR=\frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, b is not divisible by the square of a prime, and $\gcd(a,c)=1$. Find the value of $a+b+c$.

Among Us

21. **[15]** The LHS Math Team wants to play Among Us. There are so many people who want to play that they are going to form several games. Each game has at most 10 people. People are *happy* if they are in a game that has at least 8 people in it. What is the largest possible number of people who would like to play Among Us such that it is impossible to make everyone *happy*?

22. **[20]** In a game of Among Us, there are 10 players and 12 colors. Each player has a "default" color that they will automatically get if nobody else has that color. Otherwise, they get a random color that is not selected. If 10 random players with random default colors join a game one by one, the expected number of players to get their default color can be expressed as $\frac{m}{n}$. Compute $m+n$. Note that the default colors are not necessarily distinct.

23. **[20]** There are 5 people left in a game of Among Us, 4 of whom are crewmates and the last is the impostor. None of the crewmates know who the impostor is. The person with the most votes is ejected, unless there is a tie in which case no one is ejected. Each of the 5 remaining players randomly votes for someone other than themselves. The probability the impostor is ejected can be expressed as $\frac{m}{n}$. Find $m+n$.

24. **[25]** Sam has 1 Among Us task left. He and his task are located at two randomly chosen distinct vertices of a 2021-dimensional unit hypercube. Let E denote the expected distance he has to walk to get to his task, given that he is only allowed to walk along edges of the hypercube. Compute $\lceil 10E\rceil$.

Radishes

25. **[15]** Alex and Kevin are radish watching. The probability that they will see a radish within the next hour is $\frac{1}{17}$. If the probability that they will see a radish within the next 15 minutes is p, determine $\lfloor 1000p\rfloor$. Assume that the probability of seeing a radish at any given moment is uniform for the entire hour.

26. **[20]** Jeff has planted 7 radishes, labelled R, A, D, I, S, H, and E. Taiki then draws circles through S,H,I,E,D, then through E,A,R,S, and then through H,A,R,D, and notices that lines drawn through SH, AR, and ED are parallel, with $SH=ED$. Additionally, HER is equilateral, and I is the midpoint of AR. Given that $HD=2$, HE can be written as $\frac{-\sqrt{a}+\sqrt{b}+\sqrt{1+\sqrt{c}}}{2}$, where $a,b,$ and c are integers, find $a+b+c$.

27. **[25]** Ephram is growing 3 different variants of radishes in a row of 13 radishes total, but he forgot where he planted each radish variant and he can't tell what variant a radish is before he picks it. Ephram knows that he planted at least one of each radish variant, and all radishes of one variant will form a consecutive string, with all such possibilities having an equal chance of occurring. He wants to pick three radishes to bring to the farmers market, and wants them to all be of different variants. Given that he uses optimal strategy, the probability that he achieves this can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

COVID

28. **[15]** Arthur has a regular 11-gon. He labels the vertices with the letters in $CORONAVIRUS$ in consecutive order. Every non-ordered set of 3 letters that forms an isosceles triangle is a member of a set S, i.e. {C, O, R} is in S. How many elements are in S?

29. **[20]** Find the smallest possible value of n such that $n+2$ people can stand inside or on the border of a regular n-gon with side length 6 feet where each pair of people are at least 6 feet apart.

30. **[25]** A large gathering of people stand in a triangular array with 2020 rows, such that the first row has 1 person, the second row has 2 people, and so on. Every day, the people in each row infect all of the people adjacent to them in their own row. Additionally, the people at the ends of each row infect the people at the ends of the rows in front of and behind them that are at the same side of the row as they are. Given that two people are chosen at random to be infected with COVID at the beginning of day 1, what is the earliest possible day that the last uninfected person will be infected with COVID?