Theme Round Solutions

Lexington High School

December 7th, 2019

Joe Quigley

This section was written in memory of Joseph "Joe" John IV Quigley, who passed away last October. Joe Quigley ran the Math Club in Lexington for over twenty-four years, making math accessible and fun for students of all ages and abilities. His love of math was only eclipsed by his love for teaching, and he will be greatly missed by the entire Lexington community for his humor, patience, and dedication to his students.

1. Joe Quigley writes the following expression on the board for his students to evaluate:

$$2 \times 4 + 3 - 1$$

However, his students have not learned their order of operations so they randomly choose which operations to perform first, second, and third. How many different results can the students obtain?

Proposed by Alex Li

Solution. $\boxed{3}$

Evaluating all possible orderings of operation we see the only possible results are $10, 12$, and 13, so there are $\boxed{3}$ results. □

2. Joe Quigley flies airplanes around the Cartesian plane. There are two fuel stations, one at $(8,12)$ and another at $(25,5)$. He must station his home on the x-axis, and wants to be the same distance away from both station. Compute this distance.

Proposed by Ephram Chun

Solution. $\boxed{13}$

Let's call the center of the $(x,0)$. We can use the distance formula make the following equation:

$$(8-x)^2 + 12^2 = (25-x)^2 + 5^2$$

$$64 - 16x + x^2 + 144 = 625 - 50x + x^2 + 25$$

$$208 - 16x = 650 - 50x$$

$$34x = 442$$

$$x = 13$$

Therefore the center of the circle is $(0,13)$ and the radius is $5^2 + 12^2 = r^2$ which means that our answer is $\boxed{13}$. □

3. Joe Quigley has 12 students in his math class. He will distribute N worksheets among the students. Find the smallest positive integer N for which any such distribution of the N worksheets among the 12 students results in at least one student having at least 3 worksheets.

Proposed by Taiki Aiba

Solution. $\boxed{25}$

The answer is $n = \boxed{25}$. If $n \leq 24$ then it is possible for everyone to get 2 or fewer math worksheets. If $n \geq 25$, then by the Pigeonhole Principle, at least one person must have at least 3 math worksheets. □

4. Joe Quigley writes the number $4^{4^{4^4}}$ on the board. He correctly observes that this number has $2^a + b$ factors, where a and b are positive integers. Find the maximum possible value of a.

   Proposed by Kevin Zhao

   Solution. $\boxed{513}$

   We see that since $2^2 = 4$, then
   $$4^{4^{4^4}} = 4^{4^{256}} = 4^{2^{512}} = 2^{2^{513}}$$
   Because 2 is prime, then we have $2^{513} + 1$ factors, which gives $a = \boxed{513}$. □

5. Joe Quigley is teaching his students geometric series and asks them to compute the value of the following series:
   $$\sum_{x\geq 1} \frac{x(x+1)}{2^x}$$
   Compute this value.

   Proposed by Sammy Charney and Ben Epstein

   Solution. $\boxed{8}$

   Let S be the sum we are asked to compute. We now compute $2S - S = S$ by telescoping. The difference in the 2^0 term is 2, the difference in the 2^{-1} term is 4, the difference in the 2^{-2} term is 6, and this pattern continues (so $2S - S = \sum_{x\geq 1} \frac{x}{2^{x-2}}$. Now, we want to compute this sum $Y = S$. We telescope again on $2Y - Y = Y$ to get 2 in the 2^1 term, 2 in the 2^0 term, etc. Now, we have that $S = Y = \sum_{x\geq 1} \frac{1}{2^{x-3}} = \boxed{8}$. □

Astronomy

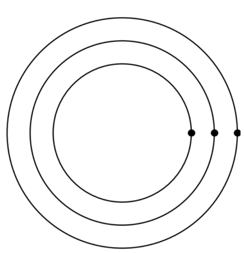
6. Alex and Anka go to see stars. Alex counts thirteen fewer stars than Anka, and both of the numbers of stars they counted are square numbers. It is known that exactly eight stars were counted by both of them. How many distinct stars did they count in total?

   Proposed by Kevin Zhao

   Solution. $\boxed{77}$

   We see that the only pair of squares such that one is greater than the other is $(36, 49)$, so using PIE, our answer is $36 + 49 - 8 = \boxed{77}$. □

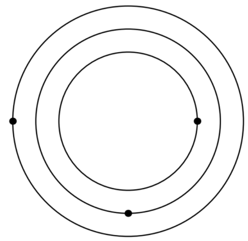
7. Three planets with coplanar, circular, and concentric orbits are shown on the backside of this page. The radii of the three circles are 3, 4, and 5. Initially, the three planets are collinear. Every hour, the outermost planet moves one-sixth of its full orbit, the middle planet moves one-fourth of its full orbit, and the innermost planet moves one-third of its full orbit (A full orbit occurs when a planet returns to its initial position). Moreover, all three planets orbit in the same direction. After three hours, what is the area of the triangle formed by the planets as its three vertices?



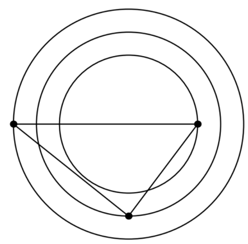
   Proposed by Taiki Aiba

Solution. $\boxed{16}$

After three hours, the positions of the three planets look like this:



After drawing the triangle, we get this:



We know that the line connecting the middle planet to the center is perpendicular to the line connecting the innermost planet to the outermost planet. We have that the base of the triangle has length $5+3=8$, and its altitude is the radius of the middle circle, or 4. Therefore, the area of the triangle with the planets as its three vertices is $\frac{1}{2}\cdot 8\cdot 4 = \boxed{16}$.

□

8. Planets X and Y are following circular orbits around the same star. It takes planet X 120 hours to complete a full trip around the star, and it takes planet Y 18 hours to complete a full trip around the star. If both planets begin at the same location in their orbit and travel clockwise, how many times will planet Y pass planet X in the time it takes planet X to complete a full trip around the star?

Proposed by Alex Li

Solution. $\boxed{5}$

In 120 hours, planet X completes one full trip around the star, and planet Y completes $\frac{20}{3} = 6 + \frac{2}{3}$ trips around the star. For the first complete trip that planet Y makes, planets X and Y are at the same location, so planet Y does not pass planet X. For each complete trip that planet Y makes after the first one, it will pass by planet X exactly one time. This means that planet Y will pass by planet X $\boxed{5}$ times. □

9. In a certain stellar system, four asteroids form a rectangle. Your spaceship lies in the rectangle formed. The distances between three of these asteroids and your space ship are 1 light minute, 8 light minutes, and 4 light minutes. If the distance between your space ship and the last asteroid is an integer number of light minutes away then how far away from the last asteroid is your space ship?

Proposed by Kaylee Ji

Solution. $\boxed{7}$

$8^2+1^2-4^2=7^2$ Nothing else yields integer values so the distance is $\boxed{7}$. □

10. Two lost lovers, Laxe and Kaan, are both standing on the equators of planets with radius 13 miles. The center of the planets are 170 miles apart. At some time, both of them are as close to each other as possible. The planets rotate in opposite directions of each other at the same rate. What is the maximum possible distance between Laxe and Kaan such that they are still able to see each other?

Proposed by Jeff Lin

Solution. $\boxed{168}$

Since both of them are on the equator, we can pretend that they are on circles of a 2-D plane. Then, we imagine them standing as close tyo each other as possible, and then imagine them rotating away from each other on the circles. The farthest they can go while still seeing each other is when they are on the points of the internal tangents, which would result in them being $2(\sqrt{85^2 - 13^2}) = \boxed{168}$ miles away. □

Holidays

11. Festivus occurs every year on December 23rd. In 2019, Festivus will occur on a Monday. On what day will Festivus occur in the year 2029?

    Proposed by Based on a proposal by Taiki Aiba

    Solution. $\boxed{\text{Sunday}}$

    Since 365 is 1 mod 7, we have that Festivus will advance one day with each year. However, we must also account for the three leap years 2020, 2024, and 2028, which will add one more advancing day. Then, Festivus will advance $10+3 = 13$ days from Monday, and 13 mod 7 is 6, so Festivus will be on a $\boxed{\text{Sunday}}$ in the year 2029. □

12. Leakey, Marpeh, Arik, and Yehau host a Secret Santa, where each one of them is assigned to give a present to somebody other than themselves. How many ways can the gifting be assigned such that everyone receives exactly one gift?

    Proposed by Jeff Lin

    Solution. $\boxed{9}$

    The derangement of 4 is $\boxed{9}$. □

13. How many permutations of the word $CHRISTMAS$ are there such that the S's are not next to each other and there is not a vowel anywhere between the two S's?

    Proposed by Taiki Aiba

    Solution. $\boxed{50400}$

    Perform casework based on how many of the other 5 consonants are in between the two S's. This yields $5\cdot 7! + 5\cdot 4\cdot 6! + 5\cdot 4\cdot 3\cdot 5! + 5\cdot 4\cdot 3\cdot 2\cdot 4! + 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 3! = \boxed{50400}$. □

14. Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, and Rudolph (9 reindeer) are guiding Santa's sleigh. They are arranged in a 3×3 array. You, the elf, have a big responsibility. You must place Santa's reindeer in a manner so that all of Santa's requests are met:

    - Donner is forgetful and must be put in the back row so Santa can keep an eye on Donner.
    - Additionally, Rudolph's big red nose distracts Donner, so Rudolph and Donner cannot be in the same column.
    - Finally, Comet is the fastest and must be put in the front row.

    How many options do you have for arranging Santa's reindeer?

    Proposed by Richard Chen

    Solution. $\boxed{34560}$

    First, we multiply by $6! = 720$ in the very end to make sure the other six reindeer on whom there are no restraints are accounted for.

    Now, we address Donner, Rudolph, and Comet:

    First, we have 3 ways in which we can put Donner. Then, we split our situation into two cases:

    - Rudolph is in the first row. Thus, we have $3\cdot 2 = 6$ total ways to arrange Rudolph and Comet, and then we have to subtract 2 because Rudolph cannot be in front of Donner. We have a total of 4 ways here.

- Rudolph is not in the first row. Thus, we have $4 \cdot 3 = 12$ total ways to put Rudolph and Comet.

In total, we have $720 \cdot 3 \cdot 16 = \boxed{34560}$ ways in which to arrange the reindeer. □

15. Marpeh has a Christmas tree in the perfect shape of a right circular cone. The tree has base radius 8 inches and slant height 32 inches. He wants to place 3 ornaments on the surface of the tree with the following rules:

    - The red ornament is placed at the top of the tree.
    - The yellow ornament is placed along the circumference of the base of the tree.
    - The blue ornament is placed such that it is the same distance from the red and yellow ornaments when traveling on the surface of the tree.

    What is the furthest possible surface distance that the blue ornament could be from the red ornament?

    Proposed by Jeff Lin

    Solution. $\boxed{16\sqrt{2}}$

    If we take the cone and unwrap it, we get a slice of a circle with radius 32 and angle 90. The red ornament would be on the center of the circle (the top of the slice), so let this point be R. Additionally, the yellow ornament could be represented as the two ends of the arc, let these be Y_1 and Y_2. Then, since the distance along the surface from the blue ornament to each of the other ornaments is equal, we know it is on either the perpendicular bisector of RY_1 or of RY_2, but we know it can't be past the angle bisector of Y_1RY_2, as shown in the diagram, so the distance is maximized when it is on the angle bisector and both perpendicular bisectors, which would give an answer of $\boxed{16\sqrt{2}}$. □