# Team Round

Lexington High School

December 7th, 2019

1. What is the smallest possible value for the product of two real numbers that differ by ten?

   *Proposed by Sooyoung Choi*

   *Solution.* $\boxed{-25}$

   Let $a-b=10$. Then $ab=b(b+10)=(b+5)^2-25$ so our answer is $\boxed{-25}$. □

2. Determine the number of positive integers $n$ with $1 \le n \le 400$ that satisfy the following:

   - $n$ is a square number.
   - $n$ is one more than a multiple of 5.
   - $n$ is even.

   *Proposed by Taiki Aiba*

   *Solution.* $\boxed{4}$

   Upon inspection, we see that $n=16$, $n=36$, $n=196$, and $n=256$ satisfy the conditions. Thus, we have that there are $\boxed{4}$ values of $n$. □

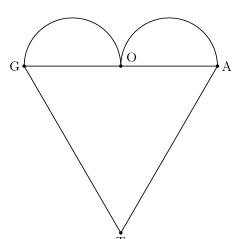
3. How many positive integers less than 2019 are either a perfect cube or a perfect square but not both?

   *Proposed by Ephram Chun*

   *Solution.* $\boxed{50}$

   We can see that there are 44 squares less than 2019 since $45^2=2025$. However some cubes are also squares, specifically when the base number is a square number before it is cubed. We can see that there are 12 cubes less than 2019 and there are only 3 square numbers less than $12: 1, 4,$ and $9$. Therefore the answer would be $44+12-3-3=\boxed{50}$. □

4. Felicia draws the heart-shaped figure $GOAT$ that is made of two semicircles of equal area and an equilateral triangle, as shown below. If $GO=2$, what is the area of the figure?




   *Proposed by Kira Tang*

   *Solution.* $\boxed{\pi+4\sqrt{3}}$

   Since $GO=2$, the radius of the semicircle is 1. Thus, the area of the semicircle with diameter $GO$ is $\frac{1}{2}\pi$. Because the two semicircles have equal area, the combined area of the two semicircles is $\pi$. The equilateral triangle $GAT$ has side length 4, so its area is $\frac{\sqrt{3}}{4}\cdot 16 = 4\sqrt{3}$. Finally, the total area of the figure is $\boxed{\pi+4\sqrt{3}}$. □

5. For distinct digits $A$, $B$, and $C$:

$$\begin{array}{ccc} & \text{A} & \text{A} \\ & \text{B} & \text{B} \\ + & \text{C} & \text{C} \\ \hline \text{A} & \text{B} & \text{C} \end{array}$$

Compute $A \cdot B \cdot C$.

*Proposed by Sammy Charney*

*Solution.* $\boxed{72}$

We see that $A$ must be 1 or 2, $B$ must be $10 - A$, and $C$ must be less than $10 - A$, so it must be less than $B$. This leads us to find $A = 1$, $B = 9$, and $C = 8$ works. $A \cdot B \cdot C = \boxed{72}$. □

6. What is the difference between the largest and smallest value for $\text{lcm}(a, b, c)$, where $a, b,$ and $c$ are distinct positive integers between 1 and 10, inclusive?

*Proposed by Jeff Lin*

*Solution.* $\boxed{626}$

The smallest is 4, and the greatest is 630, so the answer is $\boxed{626}$. □

7. Let $A$ and $B$ be points on the circumference of a circle with center $O$ such that $\angle AOB = 100°$. If $X$ is the midpoint of minor arc $\overset{\frown}{AB}$ and $Y$ is on the circumference of the circle such that $XY \perp AO$, find the measure of $\angle OBY$.

*Proposed by Alex Li*

*Solution.* $\boxed{15°}$

Let $OA$ and $BY$ intersect at $Z$. We see that $\angle BOX = 50°$, which means $\angle BYX = 25°$ since it is an inscribed angle. Thus $\angle AZY = 90° - 25° = 65°$. Thus $\angle BZO = 65°$, which gives $\angle OBY = 180° - 100° - 65° = \boxed{15°}$. □

8. When Ben works at twice his normal rate and Sammy works at his normal rate, they can finish a project together in 6 hours. When Ben works at his normal rate and Sammy works as three times his normal rate, they can finish the same project together in 4 hours. How many hours does it take Ben and Sammy to finish that project if they each work together at their normal rates?

*Proposed by Kevin Zhao*

*Solution.* $\boxed{\dfrac{60}{7}}$

Let $b$ be the number of hours Ben would take alone and $s$ be the number of hours Sammy would take alone. We see that $\frac{2}{b} + \frac{1}{s} = \frac{1}{6}$ and $\frac{1}{b} + \frac{3}{s} = \frac{1}{4}$, and then wee see multiplying the second equation gets us $\frac{2}{b} + \frac{6}{s} = \frac{1}{2}$ so

$$\left(\frac{2}{b} + \frac{6}{s}\right) - \left(\frac{2}{b} + \frac{1}{s}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3},$$

so simplifying gets us $\frac{5}{s} = \frac{1}{3}$ and thus $s = 15$. Now, plugging into the first equation gets $\frac{2}{b} = \frac{1}{10}$ so $b = 20$. We see that our answer asks for

$$\frac{1}{\frac{1}{b} + \frac{1}{s}} = \frac{1}{\frac{7}{60}} = \boxed{\frac{60}{7}}.$$

□

9. How many positive integer divisors $n$ of 20000 are there such that when 20000 is divided by $n$, the quotient is divisible by a square number greater than 1?

*Proposed by Richard Chen*

*Solution.* $\boxed{26}$

Note that the quotient is also a positive integer divisor of $n$. Then, we need to find the number of positive integer divisors $n$ of 20,000 such that $n$ is divisible by a square number greater than 1. We can factor 20,000 as $2^5 \cdot 5^4$. Then, we find that the number of divisors of 20,000 is $(5+1)(4+1) = 6 \cdot 5 = 30$. We then subtract the number of divisors that are not divisible by a square number greater than 1. We see that any divisor of 20,000 where the exponents of both 2 and 5 are less than 2 will not be divisible by a square number greater than 1. We can have the exponents of 2 and 5 be either 0 or 1, for a total of $2 \cdot 2 = 4$ divisors. Therefore, our answer is $30 - 4 = \boxed{26}$. □

10. What's the maximum number of Friday the 13th's that can occur in a year?

*Proposed by Nathan Ramesh*

*Solution.* $\boxed{3}$

We can imagine labeling every date cycling mod 7, so each residue mod 7 represents a day of the week. For simplicity, we can index January 13th as 0. Now we have the following 2 cases:

- Case 1: A leap year. Then the sequence of the 13th each month will be $0, 31 \equiv 3 \pmod 7, 3+29 \equiv 4 \pmod 7, 4+31 \equiv 0 \pmod 7, 0+30 \equiv 2 \pmod 7, 5, 0, 3, 6, 1, 4, 6$
- Case 2: No leap year. Then our sequence becomes $0, 3, 3, 6, 1, 4, 6, 2, 5, 0, 3, 5$

Since we can make Friday any index mod 7, we simply look at the frequency of the mode in each of these lists, which is at most $\boxed{3}$ □

11. Let circle $\omega$ pass through points $B$ and $C$ of triangle $ABC$. Suppose $\omega$ intersects segment $AB$ at a point $D \neq B$ and intersects segment $AC$ at a point $E \neq C$. If $AD = DC = 12$, $DB = 3$, and $EC = 8$, determine the length of $EB$.

*Proposed by Janabel Xia*

*Solution.* $\boxed{10}$

Note that $ADC$ is isosceles, so we have $\angle DAC = \angle DCA = \angle DBE$ (both subtend $\widehat{DE}$), so similarly $EB = EA$. Now setting $AE = x$, we have $x(x+8) = 12 \cdot 15$, by Power of a Point. Solving, we get $x = \boxed{10}$. □

12. Let $a, b$ be integers that satisfy the equation

$$2a^2 - b^2 + ab = 18.$$

Find the ordered pair $(a, b)$.

*Proposed by Alex Li*

*Solution.* $\boxed{(3,3),(3,0),(-3,-3),(-3,0)}$

Rewriting the expression, we can factor:

$$2a^2 - b^2 + ab = a^2 - b^2 + a^2 + ab = (a-b)(a+b) + a(a+b) = (2a-b)(a+b).$$

Note that $a = \frac{(2a-b)+(a+b)}{3}$, which means the sum of the factors must be divisible by 3. There are four possible factor pairs, $(3,6), (6,3), (-3,-6), (-6,-3)$, which gives the four solutions $\boxed{(3,3),(3,0),(-3,-3),(-3,0)}$. □

13. Let $a, b, c$ be nonzero complex numbers such that

$$a - \frac{1}{b} = 8, \quad b - \frac{1}{c} = 10, \quad c - \frac{1}{a} = 12.$$

Find $abc - \frac{1}{abc}$.

*Proposed by Ezra Erives*

*Solution.* $\boxed{990}$

Multiplying the three equations together yields $abc - \frac{1}{abc} - \left[\left(a - \frac{1}{b}\right) + \left(b - \frac{1}{c}\right) + \left(c - \frac{1}{a}\right)\right] = 960$, thus $abc - \frac{1}{abc} = 960 + (8 + 10 + 12) = \boxed{990}$. □

14. Let $\Delta ABC$ be an equilateral triangle of side length 1. Let $\omega_0$ be the incircle of $\Delta ABC$, and for $n > 0$, define the infinite progression of circles $\omega_n$ as follows:

    - $\omega_n$ is tangent to $AB$ and $AC$ and externally tangent to $\omega_{n-1}$.
    - The area of $\omega_n$ is strictly less than the area of $\omega_{n-1}$.

    Determine the total area enclosed by all $\omega_i$ for $i \geq 0$.

    *Proposed by Richard Chen*

    *Solution.* $\boxed{\frac{3\pi}{32}}$

    We label three more points: $H$, the altitude from $A$ to $BC$, $I$, the incenter of $\triangle ABC$, and $J$, the other intersection of the incircle and $AI$.

    Quick calculation gives that $AH = \frac{\sqrt{3}}{2}$, and, because $I$ is also the centroid the ratio of $AI$ to $IH$ is $2:1$ and the inradius is $\frac{\sqrt{3}}{6}$, so the area of $\omega_1 = \frac{\pi}{12}$.

    Therefore, if we draw the line through $J$ parallel to $BC$ and let the intersection point with $AB$ be $X$ and the intersection point with $AC$ be $Y$, we see that $\triangle AXY \sim \triangle ABC$. Additionally, because $AJ$ is $\frac{1}{3}$ of $AH$, we know that the proportions of $\triangle AXY$ are scaled down by a factor of 3, and therefore $\omega_2$ has an area $\frac{1}{9}$ that of $\omega_1$.

    We can see that the same property will hold for $\omega_3, \omega_4$, and so on: $[\omega_n] = \left(\frac{1}{9}\right)^{n-1} \cdot \frac{\pi}{12}$. Summing up, the total area would be $\left(1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 \cdots\right) \cdot \frac{\pi}{12} = \boxed{\frac{3\pi}{32}}$. □

15. Determine the remainder when $13^{2020} + 11^{2020}$ is divided by 144.

    *Proposed by Jeff Lin*

    *Solution.* $\boxed{2}$

    We write this as $(12+1)^{2020} + (12-1)^{2020}$. Expanding and ignoring all of the terms with a $12^2$, we get $2020 \cdot 1^{2019} \cdot 12 + 1^{2020} - 2020 \cdot 1^{2019} \cdot 12 + 1^{2020} = \boxed{2}$. □

16. Let $x$ be a solution to $x + \frac{1}{x} = 1$. Compute $x^{2019} + \frac{1}{x^{2019}}$.

    *Proposed by*

    *Solution.* $\boxed{-2}$

    Solution 1: Let $T_k(y) = y^k + \frac{1}{y^k}$. It follows that $T_{k+1}(y) = T_k(y)(y + \frac{1}{y}) - T_{k-1}(y)$ and so $T_{k+1}(x) = T_k(x) - T_{k-1}(x)$. Starting from $T_1(x) = 1$ and $T_2(x) = T_1^2 - 2 = -1$, we find that $T_{k+6}(x) = T_k(x)$, and so $T_{2019}(x) = T_3(x) = \boxed{-2}$. □

17. The positive integers are colored black and white such that if $n$ is one color, then $2n$ is the other color. If all of the odd numbers are colored black, then how many numbers between 100 and 200 inclusive are colored white?

    *Proposed by Ezra Erives*

    *Solution.* $\boxed{34}$

    We want to find the number of numbers that are either $2 \pmod 4$, $8 \pmod{16}$, $32 \pmod{64}$, or $128 \pmod{256}$, which are the only numbers at most 200 that are an odd number of 2s times an odd number. The first numbers range from $102-198$ every 4, of which there are 25. The second numbers range from $104-200$ every 16, of which there are 7. The only number satisfying the third condition is 160 and the only number satisfying the fourth condition is 128. This gives a total of $25+7+1+1 = \boxed{34}$ white numbers. □

18. What is the expected number of rolls it will take to get all six values of a six-sided die face-up at least once?

    *Proposed by Anka Hu*

*Solution.* $\boxed{\dfrac{147}{10}}$

The first roll will always result in a never-before-rolled value, or with a probability $\frac{6}{6}$. This will take 1 roll.

The next roll has a $\frac{5}{6}$ probability of resulting in a new value, and after that there is a $\frac{4}{6}$ probability that the next roll will result in the third new value. Therefore, the expected number of rolls is

$$1+\frac{6}{5}+\ldots+6=6\sum_{i=1}^{6}\frac{1}{i}=\boxed{\frac{147}{10}}.$$

□

19. Let $\Delta ABC$ have side lengths $AB = 19, BC = 2019$, and $AC = 2020$. Let $D, E$ be the feet of the angle bisectors drawn from $A$ and $B$, and let $X, Y$ to be the feet of the altitudes from $C$ to $AD$ and $C$ to $BE$, respectively. Determine the length of $XY$.

*Proposed by Jeff Lin*

*Solution.* $\boxed{2010}$

We reflect $C$ over $X$ and $Y$. These points are clearly on $AB$ because we are reflecting $C$ over the angle bisectors. Call these reflection points $C'$ and $C''$. We can clearly see that $AC' = 2020$ and $BC'' = 2019$. Since $AB = 19$, we get $C'C'' = 4020$. Since $X$ and $Y$ are midpoints of $CC'$ and $CC''$, we get $XY = \frac{C'C''}{2} = \boxed{2010}$. □

20. Suppose I have 5 unit cubes of cheese that I want to divide evenly amongst 3 hungry mice. I can cut the cheese into smaller blocks, but cannot combine blocks into a bigger block. Over all possible choices of cuts in the cheese, what's the largest possible volume of the smallest block of cheese?

*Proposed by Jeremy Zhou*

*Solution.* $\boxed{\dfrac{5}{12}}$

Each mice should get a volume of $\frac{5}{3}$ of cheese total. We see that we must make at least one cut in the cheese, so we have an upper bound of $\frac{1}{2}$. Trying around, we quickly see that we can achieve $\frac{1}{3}$ by leaving 3 blocks untouched, and cutting the other two blocks into two blocks each of size $\frac{1}{3}$ and $\frac{2}{3}$.

Suppose we wish to achieve larger than $\frac{1}{3}$. Note that if we leave any cube uncut, then the mouse we give that cube to still needs $\frac{2}{3}$. If this $\frac{2}{3}$ comes from 1 unit cube, then the leftover block(s) from that unit cube are at most $\frac{1}{3}$. If it comes from 2 or more blocks of cheese, then the smallest of those blocks is at most $\frac{1}{3}$. Therefore, we must cut each of the 5 unit cubes at least once. This yields at least 10 blocks of cheese, so Pigeonhole Principle tells us that one mouse gets at least 4 blocks of cheese. Since these 4 blocks have total volume $\frac{5}{3}$, the smallest of these blocks is at most $\frac{5}{12}$.

All that remains is to provide a construction for $\frac{5}{12}$: Suppose we cut 4 of our cubes into $\frac{7}{12}$ and $\frac{5}{12}$, and the last into $\frac{1}{2}$ and $\frac{1}{2}$. Then we give one mouse 4 of the $\frac{5}{12}$ blocks, and the other two each get $2\cdot\frac{7}{12}+\frac{1}{2}=\frac{5}{3}$, so indeed our answer is $\boxed{\dfrac{5}{12}}$.

□