Team Round

Lexington High School

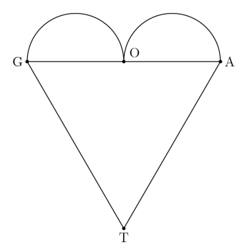
December 7th, 2019

1. What is the smallest possible value for the product of two real numbers that differ by ten?

2. Determine the number of positive integers n with $1 \le n \le 400$ that satisfy the following:
   - n is a square number.
   - n is one more than a multiple of 5.
   - n is even.

3. How many positive integers less than 2019 are either a perfect cube or a perfect square but not both?

4. Felicia draws the heart-shaped figure $GOAT$ that is made of two semicircles of equal area and an equilateral triangle, as shown below. If $GO = 2$, what is the area of the figure?



5. For distinct digits A, B, and C:

$$\begin{array}{ccc} & \mathrm{A} & \mathrm{A} \\ & \mathrm{B} & \mathrm{B} \\ + & \mathrm{C} & \mathrm{C} \\ \hline \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array}$$

   Compute $A \cdot B \cdot C$.

6. What is the difference between the largest and smallest value for $\text{lcm}(a,b,c)$, where $a,b,$ and c are distinct positive integers between 1 and 10, inclusive?

7. Let A and B be points on the circumference of a circle with center O such that $\angle AOB = 100^\circ$. If X is the midpoint of minor arc $\overset{\frown}{AB}$ and Y is on the circumference of the circle such that $XY \perp AO$, find the measure of $\angle OBY$.

8. When Ben works at twice his normal rate and Sammy works at his normal rate, they can finish a project together in 6 hours. When Ben works at his normal rate and Sammy works as three times his normal rate, they can finish the same project together in 4 hours. How many hours does it take Ben and Sammy to finish that project if they each work together at their normal rates?

9. How many positive integer divisors n of 20000 are there such that when 20000 is divided by n, the quotient is divisible by a square number greater than 1?

10. What's the maximum number of Friday the 13th's that can occur in a year?

11. Let circle ω pass through points B and C of triangle ABC. Suppose ω intersects segment AB at a point $D \ne B$ and intersects segment AC at a point $E \ne C$. If $AD = DC = 12$, $DB = 3$, and $EC = 8$, determine the length of EB.

12. Let a, b be integers that satisfy the equation
$$2a^2 - b^2 + ab = 18.$$
Find the ordered pair (a, b).

13. Let a, b, c be nonzero complex numbers such that
$$a - \frac{1}{b} = 8, \quad b - \frac{1}{c} = 10, \quad c - \frac{1}{a} = 12.$$
Find $abc - \frac{1}{abc}$.

14. Let ΔABC be an equilateral triangle of side length 1. Let ω_0 be the incircle of ΔABC, and for $n > 0$, define the infinite progression of circles ω_n as follows:
    - ω_n is tangent to AB and AC and externally tangent to ω_{n-1}.
    - The area of ω_n is strictly less than the area of ω_{n-1}.

    Determine the total area enclosed by all ω_i for $i \geq 0$.

15. Determine the remainder when $13^{2020} + 11^{2020}$ is divided by 144.

16. Let x be a solution to $x + \frac{1}{x} = 1$. Compute $x^{2019} + \frac{1}{x^{2019}}$.

17. The positive integers are colored black and white such that if n is one color, then $2n$ is the other color. If all of the odd numbers are colored black, then how many numbers between 100 and 200 inclusive are colored white?

18. What is the expected number of rolls it will take to get all six values of a six-sided die face-up at least once?

19. Let ΔABC have side lengths $AB = 19, BC = 2019$, and $AC = 2020$. Let D, E be the feet of the angle bisectors drawn from A and B, and let X, Y to be the feet of the altitudes from C to AD and C to BE, respectively. Determine the length of XY.

20. Suppose I have 5 unit cubes of cheese that I want to divide evenly amongst 3 hungry mice. I can cut the cheese into smaller blocks, but cannot combine blocks into a bigger block. Over all possible choices of cuts in the cheese, what's the largest possible volume of the smallest block of cheese?