Individual Round

Lexington High School

December 7th, 2019

1. For positive real numbers x, y, the operation $\otimes$ is given by $x \otimes y = \sqrt{x^2 - y}$ and the operation $\oplus$ is given by $x \oplus y = \sqrt{x^2 + y}$. Compute
$$(((5 \otimes 4) \oplus 3) \otimes 2) \oplus 1.$$

2. Janabel is cutting up a pizza for a party. She knows there will either be 4, 5, or 6 people at the party including herself, but can't remember which. What is the least number of slices Janabel can cut her pizza to guarantee that everyone at the party will be able to eat an equal number of slices?

3. If the numerator of a certain fraction is added to the numerator and the denominator, the result is $\frac{20}{19}$. What is the fraction?

4. Let trapezoid $ABCD$ be such that $AB||CD$. Additionally, $AC = AD = 5$, $CD = 6$, and $AB = 3$. Find BC.

5. At Merrick's Ice Cream Parlor, customers can order one of three flavors of ice cream and can have their ice cream in either a cup or a cone. Additionally, customers can choose any combination of the following three toppings: sprinkles, fudge, and cherries. How many ways are there to buy ice cream?

6. Find the minimum possible value of the expression $|x+1| + |x-4| + |x-6|$.

7. How many 3 digit numbers have an even number of even digits?

8. Given that the number $1a99b67$ is divisible by 7, 9, and 11, what are a and b? Express your answer as an ordered pair.

9. Let O be the center of a quarter circle with radius 1 and $\widehat{AB}$ be the quarter of the circle's circumference. Let M, N be the midpoints of AO and BO, respectively. Let X be the intersection of AN and BM. Find the area of the region enclosed by $\widehat{AB}, AX, BX$.

10. Each square of a 5-by-1 grid of squares is labeled with a digit between 0 and 9, inclusive, such that the sum of the numbers on any two adjacent squares is divisible by 3. How many such labelings are possible if each digit can be used more than once?

11. A two-digit number has the property that the difference between the number and the sum of its digits is divisible by the units digit. If the tens digit is 5, how many different possible values of the units digit are there?

12. There are 2019 red balls and 2019 white balls in a jar. One ball is drawn and replaced with a ball of the other color. The jar is then shaken and one ball is chosen. What is the probability that this ball is red?

13. Let $ABCD$ be a square with side length 2. Let ℓ denote the line perpendicular to diagonal AC through point C, and let E and F be the midpoints of segments BC and CD, respectively. Let lines AE and AF meet ℓ at points X and Y, respectively. Compute the area of $\triangle AXY$.

14. Express $\sqrt{21 - 6\sqrt{6}} + \sqrt{21 + 6\sqrt{6}}$ in simplest radical form.

15. Let $\triangle ABC$ be an equilateral triangle with side length two. Let D and E be on AB and AC respectively such that $\angle ABE = \angle ACD = 15°$. Find the length of DE.

16. 2018 ants walk on a line that is 1 inch long. At integer time t seconds, the ant with label $1 \le t \le 2018$ enters on the left side of the line and walks among the line at a speed of $\frac{1}{t}$ inches per second, until it reaches the right end and walks off. Determine the number of ants on the line when $t = 2019$ seconds.

17. Determine the number of ordered tuples $(a_1, a_2, \ldots, a_5)$ of positive integers that satisfy $a_1 \le a_2 \le \cdots \le a_5 \le 5$.

18. Find the sum of all positive integer values of k for which the equation

$$\gcd(n^2 - n - 2019, n + 1) = k$$

has a positive integer solution for n.

19. Let $a_0 = 2$, $b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = 2a_n + b_n + 1,$$
$$b_{n+1} = a_n + 2b_n - 1.$$

Find the remainder when a_{2019} is divided by 100.

20. In ΔABC, let AD be the angle bisector of $\angle BAC$ such that D is on segment BC. Let T be the intersection of ray $\overrightarrow{CB}$ and the line tangent to the circumcircle of $\triangle ABC$ at A. Given that $BD = 2$ and $TC = 10$, find the length of AT.