Guts Round

Lexington High School

December 7th, 2019

10th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** A positive integer is said to be *transcendent* if it leaves a remainder of 1 when divided by 2. Find the 1010th smallest positive integer that is transcendent.

______________ 2. **[5]** The two diagonals of a square are drawn, forming four triangles. Determine, in degrees, the sum of the interior angle measures in all four triangles.

______________ 3. **[5]** Janabel multiplied 2 two-digit numbers together and the result was a four digit number. If the thousands digit was nine and hundreds digit was seven, what was the tens digit?

10th Annual Lexington Math Tournament - Guts Round - Part 2

Team Name: ______________________________

______________ 4. **[5]** Two friends, Arthur and Brandon, are comparing their ages. Arthur notes that 10 years ago, his age was a third of Brandon's current age. Brandon points out that in 12 years, his age will be double of Arthur's current age. How old is Arthur now?

______________ 5. **[5]** A farmer makes the observation that gathering his chickens into groups of 2 leaves 1 chicken left over, groups of 3 leaves 2 chickens left over, and groups of 5 leaves 4 chickens left over. Find the smallest possible number of chickens that the farmer could have.

______________ 6. **[5]** Charles has a bookshelf with 3 layers and 10 indistinguishable books to arrange. If each layer must hold less books than the layer below it and a layer cannot be empty, how many ways are there for Charles to arrange his 10 books?

10th Annual Lexington Math Tournament - Guts Round - Part 3

Team Name: ______________________________

______________ 7. **[6]** Determine the number of factors of 2^{2019}.

______________ 8. **[6]** The points A, B, C, and D lie along a line in that order. It is given that $\overline{AB} : \overline{CD} = 1 : 7$ and $\overline{AC} : \overline{BD} = 2 : 5$. If $BC = 3$, find AD.

______________ 9. **[6]** A positive integer n is equal to one-third the sum of the first n positive integers. Find n.

10th Annual Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[6]** Let the numbers $a, b, c,$ and d be in arithmetic progression. If $a+2b+3c+4d = 5$ and $a = \frac{1}{2}$, find $a+b+c+d$.

______________ 11. **[6]** Ten people playing brawl stars are split into five duos of 2. Determine the probability that Jeff and Ephram are paired up.

______________ 12. **[6]** Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = \left\lceil \frac{F_{n-1}+F_{n-2}}{2} \right\rceil + 1$, where $\lceil r \rceil$ denotes the least integer greater than or equal to r. Find F_{2019}.

10th Annual Lexington Math Tournament - Guts Round - Part 5

Team Name: ______________________________

______________ 13. **[7]** Determine the number of different circular bracelets can be made with 7 beads, all either colored red or black.

______________ 14. **[7]** The product of 260 and n is a perfect square. The 2020th least possible positive integer value of n can be written as $p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot p_4^{e_4}$. Find the sum $p_1 + p_2 + p_3 + p_4 + e_1 + e_2 + e_3 + e_4$.

______________ 15. **[7]** Let B and C be points along the circumference of circle ω. Let A be the intersection of the tangents at B and C and let $D \neq A$ be on $\overrightarrow{AC}$ such that $AC = CD = 6$. Given $\angle BAC = 60°$, find the distance from point D to the center of ω.

10th Annual Lexington Math Tournament - Guts Round - Part 6

Team Name: ______________________________

______________ 16. **[7]** Evaluate $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$.

______________ 17. **[7]** Let $n(A)$ be the number of elements of set A and $||A||$ be the number of subsets of set A. Given that $||A|| + 2||B|| = 2^{2020}$, find the value of $n(B)$.

______________ 18. **[7]** a and b are positive integers and $8^a 9^b$ has 578 factors. Find ab.

10th Annual Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

______________ 19. **[8]** Determine the probability that a randomly chosen positive integer is relatively prime to 2019.

______________ 20. **[8]** A 3-by-3 grid of squares is to be numbered with the digits 1 through 9 such that each number is used once and no two even-numbered squares are adjacent. Determine the number of ways to number the grid.

______________ 21. **[8]** In ΔABC, point D is on AC so that $\frac{AD}{DC} = \frac{1}{13}$. Let point E be on BC, and let F be the intersection of AE and BD. If $\frac{DF}{FB} = \frac{2}{7}$ and the area of ΔDBC is 26, compute the area of ΔFAB.

10th Annual Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[8]** A quarter circle with radius 1 is located on a line with its horizontal base on the line and to the left of the vertical side. It is then rolled to the right until it reaches its original orientation. Determine the distance traveled by the center of the quarter circle.

______________ 23. **[8]** In 1734, mathematician Leonhard Euler proved that $\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$. With this in mind, calculate the value of $\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$ (the series obtained by negating every other term of the original series).

______________ 24. **[8]** Billy the biker is competing in a bike show where he can do a variety of tricks. He knows that one trick is worth 2 points, 1 trick is worth 3 points, and 1 is worth 5 points, but he doesn't remember which trick is worth what amount. When it's Billy's turn to perform, he does 6 tricks, randomly choosing which trick to do. Compute the sum of all the possible values of points that Billy could receive in total.

10th Annual Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

______________ 25. **[9]** Find the largest prime factor of 1031301.

______________ 26. **[9]** Let $ABCD$ be a trapezoid such that $AB \parallel CD, \angle ABC = 90°, AB = 5, BC = 20, CD = 15$. Let X, Y be the intersection of the circle with diameter BC and segment AD. Find the length of XY.

______________ 27. **[9]** A string consisting of 1's, 2's, and 3's is said to be a superpermutation of the string 123 if it contains every permutation of 123 as a contiguous substring. Find the smallest possible length of such a superpermutation.

10th Annual Lexington Math Tournament - Guts Round - Part 10

Team Name: ______________________________

______________ 28. **[11]** Suppose that we have a function $f(x) = x^3 - 3x^2 + 3x$, and for all $n \geq 1$, $f^n(x)$ is defined by the function f applied n times to x. Find the remainder when $f^5(2019)$ is divided by 100.

______________ 29. **[11]** A function $f : \{1,2,\dots,10\} \to \{1,2,\dots,10\}$ is said to be *happy* if it is a bijection and for all $n \in \{1,2,\dots,10\}$, $|n - f(n)| \leq 1$. Compute the number of happy functions.

______________ 30. **[11]** Let ΔLMN have side lengths $LM = 15$, $MN = 14$, and $NL = 13$. Let the angle bisector of $\angle MLN$ meet the circumcircle of ΔLMN at a point $T \neq L$. Determine the area of ΔLMT.

10th Annual Lexington Math Tournament - Guts Round - Part 11

Team Name: ______________________________

______________ 31. **[13]** Find the value of

$$\sum_{d|2200} \tau(d),$$

where $\tau(n)$ denotes the number of divisors of n, and where $a|b$ means that $\frac{b}{a}$ is a positive integer.

______________ 32. **[13]** Let complex numbers $\omega_1, \omega_2, \dots, \omega_{2019}$ be the solutions to the equation $x^{2019} - 1 = 0$. Evaluate

$$\sum_{i=1}^{2019} \frac{1}{1+\omega_i}$$

______________ 33. **[13]** Let M be a nonnegative real number such that $x^{x^{x^{\cdots}}}$ diverges for all $x > M$, and $x^{x^{x^{\cdots}}}$ converges for all $0 < x \leq M$. Find M.

10th Annual Lexington Math Tournament - Guts Round - Part 12

Team Name: ________________________________

________________ 34. **[15]** Estimate the number of digits in $\binom{2019}{1009}$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

________________ 35. **[15]** You may submit any integer E from 1 to 30. Out of the teams that submit this problem, your score will be

$$\frac{E}{2(\text{the number of teams who chose } E)}.$$

________________ 36. **[15]** We call a $m \times n$ *domino-tiling* a configuration of 2×1 dominoes on an $m \times n$ cell grid such that each domino occupies exactly 2 cells of the grid and all cells of the grid are covered. How many 8×8 domino-tilings are there? If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$