Individual Round

Lexington High School

April 7, 2018

1. Evaluate $6^4 + 5^4 + 3^4 + 2^4$.

2. What digit is most frequent between 1 and 1000 inclusive?

3. Let $n = \gcd(2^2 \cdot 3^3 \cdot 4^4, 2^4 \cdot 3^3 \cdot 4^2)$. Find the number of positive integer factors of n.

4. Suppose p and q are prime numbers such that $13p + 5q = 91$. Find $p + q$.

5. Let $x = (5^3 - 5)(4^3 - 4)(3^3 - 3)(2^3 - 2)(1^3 - 1)$. Evaluate 2018^x.

6. Liszt the lister lists all 24 four-digit integers that contain each of the digits $1, 2, 3, 4$ exactly once in increasing order. What is the sum of the 20th and 18th numbers on Liszt's list?

7. Square $ABCD$ has center O. Suppose M is the midpoint of AB and $OM + 1 = OA$. Find the area of square $ABCD$.

8. How many positive 4-digit integers have at most 3 distinct digits?

9. Find the sum of all distinct integers obtained by placing $+$ and $-$ signs in the following spaces

$$2\_3\_4\_5.$$

10. In triangle $ABC, \angle A = 2\angle B$. Let I be the intersection of the angle bisectors of B and C. Given that $AB = 12, BC = 14$, and $CA = 9$, find AI.

11. You have a $3 \times 3 \times 3$ cube in front of you. You are given a knife to cut the cube and you are allowed to move the pieces after each cut before cutting it again. What is the minimum number of cuts you need to make in order to cut the cube into 27 $1 \times 1 \times 1$ cubes?

12. How many ways can you choose 3 distinct numbers from the set $\{1, 2, 3, \ldots, 20\}$ to create a geometric sequence?

13. Find the sum of all multiples of 12 that are less than 10^4 and contain only 0 and 4 as digits.

14. What is the smallest positive integer that has a different number of digits in each base from 2 to 5?

15. Given 3 real numbers (a, b, c) such that

$$\frac{a}{b+c} = \frac{b}{3a+3c} = \frac{c}{a+3b},$$

find all possible values of

$$\frac{a+b}{c}.$$

16. Let S be the set of lattice points (x, y, z) in $\mathbb{R}^3$ satisfying $0 \le x, y, z \le 2$. How many distinct triangles exist with all three vertices in S?

17. Let $\oplus$ be an operator such that for any 2 real numbers a and b, $a \oplus b = 20ab - 4a - 4b + 1$. Evaluate

$$\frac{1}{10} \oplus \frac{1}{9} \oplus \frac{1}{8} \oplus \frac{1}{7} \oplus \frac{1}{6} \oplus \frac{1}{5} \oplus \frac{1}{4} \oplus \frac{1}{3} \oplus \frac{1}{2} \oplus 1.$$

18. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(f(x)) = x$ and $f(2f(2x+16)) = f\left(\frac{1}{x+8}\right)$ for all positive integers x. Find $f(2018)$.

19. There exists an integer divisor d of 240100490001 such that $490000 < d < 491000$. Find d.

20. Let a and b be not necessarily distinct positive integers chosen independently and uniformly at random from the set $\{1,2,3,\cdots,511,512\}$. Let $x = \frac{a}{b}$. Find the probability that $(-1)^x$ is a real number.

21. In $\triangle ABC$ we have $AB = 4, BC = 6$, and $\angle ABC = 135°$. $\angle ABC$ is trisected by rays B_1 and B_2. Ray B_1 intersects side CA at point F, and ray B_2 intersects side CA at point G. What is the area of $\triangle BFG$?

22. A level number is a number which can be expressed as $x \cdot \lfloor x \rfloor \cdot \lceil x \rceil$ where x is a real number. Find the number of positive integers less than or equal to 1000 which are also level numbers.

23. Triangle $\triangle ABC$ has sidelengths $AB = 13, BC = 14, CA = 15$ and circumcenter O. Let D be the intersection of AO and BC. Compute BD/DC.

24. Let $f(x) = x^4 - 3x^3 + 2x^2 + 5x - 4$ be a quartic polynomial with roots a, b, c, d. Compute

$$\left(a+1+\frac{1}{a}\right)\left(b+1+\frac{1}{b}\right)\left(c+1+\frac{1}{c}\right)\left(d+1+\frac{1}{d}\right).$$

25. Triangle $\triangle ABC$ has centroid G and circumcenter O. Let D be the foot of the altitude from A to BC. If $AD = 2018, BD = 20$, and $CD = 18$, find the area of triangle $\triangle DOG$.