Individual Round Solutions

Lexington High School

April 8, 2017

1. Find the number of zeroes at the end of 20^{17}.

   Proposed by: Peter Rowley

   ANSWER: $\boxed{17}$

   SOLUTION: $20^{17} = 2^{17} \cdot 10^{17}$ which has 17 trailing zeroes.

2. Express $\frac{1}{\sqrt{20}+\sqrt{17}}$ in simplest radical form.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{\dfrac{2\sqrt{5}-\sqrt{17}}{3}}$

   SOLUTION:

3. John draws a square $ABCD$. On side AB he draws point P so that $\frac{BP}{PA} = \frac{1}{20}$ and on side BC he draws point Q such that $\frac{BQ}{QC} = \frac{1}{17}$. What is the ratio of the area of $\triangle PBQ$ to the area of $ABCD$?

   Proposed by: Evan Fang

   ANSWER: $\boxed{\dfrac{1}{756}}$

   SOLUTION: Let s be the sidelength of the square. Then the area of $\triangle BPQ$ is $\frac{1}{18} \cdot \frac{1}{21} \cdot \frac{1}{2} \cdot s^2$, so the answer is $\frac{1}{21 \cdot 18 \cdot 2} = \frac{1}{756}$.

4. Alfred, Bill, Clara, David, and Emily are sitting in a row of five seats at a movie theater. Alfred and Bill don't want to sit next to each other, and David and Emily have to sit next to each other. How many arrangements can they sit in that satisfy these constraints?

   Proposed by: Peter Rowley

   ANSWER: $\boxed{8}$

   SOLUTION: Note that Alfred has to sit only next to David and Emily but they have to sit next to each other, so Alfred is on one end. If Alfred is first, David and Emily are next in either order, which leaves Bill and Clara in either order for $2 \times 2 = 4$ arrangements. If Alfred is last, we have a reflection of these cases, giving another 4 arrangements. This gives a total of 8 arrangements.

5. Alex is playing a game with an unfair coin which has a $\frac{1}{5}$ chance of flipping heads and a $\frac{4}{5}$ chance of flipping tails. He flips the coin three times and wins if he flipped at least one head and one tail. What is the probability that Alex wins?

   Proposed by: Evan Fang

ANSWER: $\boxed{\dfrac{12}{25}}$

SOLUTION: The probability he loses is if he flips all three times and it lands on one side each time. This happens with $(\frac{1}{5})^3 + (\frac{4}{5})^3 = \frac{65}{125} = \frac{13}{25}$. So the probability he wins is $1 - \frac{13}{25} = \frac{12}{25}$

6. Positive two-digit number $\overline{ab}$ has 8 divisors. Find the number of divisors of the four-digit number $\overline{abab}$.

Proposed by: John Guo

ANSWER: $\boxed{16}$

SOLUTION: First, note that $\overline{abab}$ can be rewritten as:

$\overline{abab} = 100 \times \overline{ab} + \overline{ab} = 101 \times \overline{ab}$.

Since $\overline{ab}$ is a two-digit number, there is no way for $\overline{ab}$ to have a factor of 101 in its prime factorization, as 101 is a prime and greater than any two-digit number.

Thus, $\overline{abab}$ has two kinds of factors: factors of $\overline{ab}$ and factors of $\overline{ab}$ multiplied by 101. It follows that the number of factors of $\overline{abab} = 2 \cdot (\text{number of factors of} \overline{ab}) = 8 \times 2 = 16$.

7. Call a positive integer n *diagonal* if the number of diagonals of a convex n-gon is a multiple of the number of sides. Find the number of diagonal positive integers less than or equal to 2017.

Proposed by: John Guo

ANSWER: $\boxed{1008}$

SOLUTION: First, the number of diagonals of a convex n-gon is $\dfrac{n(n-3)}{2}$.

Note that this expression will only be a multiple of n if n is odd, as if n is even the 2 in the denominator prevents the expression from having n as one of its factors. Furthermore, note that the expression will always be a multiple of n if it is odd, as the expression always produces a whole number.

We then count the number of odd numbers between 3 and 2017 (as there is no n-gon with 1 side):

$1 + \dfrac{2017-3}{2} = 1008$

8. There are 4 houses on a street, with 2 on each side, and each house can be colored one of 5 different colors. Find the number of ways that the houses can be painted such that no two houses on the same side of the street are the same color and not all the houses are different colors.

Proposed by: Andrew Shen

ANSWER: $\boxed{280}$

SOLUTION: Choose the colors of the houses on one side in $5 \cdot 4 = 20$ ways. Complimentary counting the number of ways to color the other side, there are $20 - 3 \cdot 2 = 14$ ways. In total there are $20 \cdot 14 = 280$ colorings.

9. Compute

$$|2017 - |2016 - |2015 - |\cdots|3 - |2 - 1||\cdots||||.$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{1009}$

SOLUTION: Let x_n be the result when 2017 is replaced with n. Then $x_n = |n - x_{n-1}|$. It's easy to show by induction that $x_n = \lceil \frac{n}{2} \rceil$. For $n = 2017$, this is 1009.

10. Given points A, B in the coordinate plane, let $A \oplus B$ be the unique point C such that $\overline{AC}$ is parallel to the x-axis and $\overline{BC}$ is parallel to the y-axis. Find the point (x, y) such that

$$((x,y) \oplus (0,1)) \oplus (1,0) = (2016, 2017) \oplus (x, y).$$

Proposed by: Evan Fang

ANSWER: $\boxed{(1,2017)}$

SOLUTION: First, notice that $(x_1, y_1) \oplus (x_2, y_2) = (x_2, y_1)$. This can be easily seen by drawing a graph

So $((x,y) \oplus (0,1)) \oplus (1,0) = (0,y) \oplus (1,0) = (1,y) = (2016,2017) \oplus (x,y) = (x,2017)$

So our answer is $(1,2017)$

11. In the following subtraction problem, different letters represent different nonzero digits.

$$\begin{array}{r} \text{MATH} \\ -\quad \text{HAM} \\ \hline \text{LMT} \end{array}$$

How many ways can the letters be assigned values to satisfy the subtraction problem?

Proposed by: Peter Rowley

ANSWER: $\boxed{4}$

SOLUTION: $MATH$ has 4 digits and HAM, LMT have 3 digits, so $M = 1$. As $H \neq 0$, we have $H - 1 = T$ and $T - A = 1$ from the last two columns. Thus, we know that $H - A = 2$, so $10 + A - H = 8$ and $L = 8$. This means that (A, T, H) are three consecutive increasing one digit integers, none of which are 1 or 8. This means that A can be anywhere from 2 to 5, giving 4 options.

12. If m and n are integers such that $17n + 20m = 2017$, then what is the minimum possible value of $|m - n|$?

Proposed by: Angela Gong, Nathan Ramesh

ANSWER: $\boxed{12}$

SOLUTION: 2017 has a remainder of 1 when divided by 3, as does 4, so m must also have a remainder of 1 when divided by 3. This gives $(m, n) = (1, 671)$, so all (m, n) are of the form $(1 + 3k, 671 - 4k)$. We want to make m and n as close as possible to $\frac{2017}{7} = 288\frac{1}{7}$. The two closest pairs to this are $(286, 291)$ and $(289, 287)$, of which the second has a smaller difference, so the minimum value is 2.

13. Let $f(x) = x^4 - 3x^3 + 2x^2 + 7x - 9$. For some complex numbers a, b, c, d, it is true that $f(x) = (x^2 + ax + b)(x^2 + cx + d)$ for all complex numbers x. Find $\frac{a}{b} + \frac{c}{d}$.

Proposed by: Nathan Ramesh and Janabel Xia

ANSWER: $\boxed{-\frac{7}{9}}$

SOLUTION:

14. A positive integer is called an *imposter* if it can be expressed in the form $2^a + 2^b$ where a, b are non-negative integers and $a \neq b$. How many almost positive integers less than 2017 are imposters?

Proposed by: Evan Fang

ANSWER: $\boxed{55}$

SOLUTION: An integer is an almost power of 2 if and only if there are exactly two 1s written in its binary expression. $2017 = 11111100001_2$. So Imagine we had a string of 11 0s and then chose 2 of the 0s to be 1. This counts exactly the number of almost powers of 2 less than 2017. Thus the answer is $\binom{11}{2} = 55$

15. Evaluate the infinite sum
$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2^{n+1}} = \frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32} + \cdots.$$

Proposed by: Albert Zhang

ANSWER: $\boxed{4}$

SOLUTION: The sum is
$$\frac{1}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)^3 = \frac{1}{2} \cdot 8 = 4.$$
You can check this with the so-called "stars and bars" in the following way. Let $f(x) = 1 + x + x^2 + x^3 + \cdots$. Then the coefficient of x^n in $f(x)^3$ is the number of triples (a, b, c) such that $a + b + c = n$, which is well known to be $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$. When $x = \frac{1}{2}$ terms are of the form $\frac{n(n+1)}{2^n}$. This is why we multiply by $\frac{1}{2}$ at the start.

16. Each face of a regular tetrahedron is colored either red, green, or blue, each with probability $\frac{1}{3}$. What is the probability that the tetrahedron can be placed with one face down on a table such that each of the three visible faces are either all the same color or all different colors?

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{7}{9}}$

SOLUTION: We'll find the probability of the complement, that is, we'll find the probability that there aren't three adjacent faces with different colors and there aren't three adjacent faces with the same colors. Any three of the four faces of a tetrahedron are adjacent, so the tetrahedron must be colored with at most two colors. At the same time, no two of the used colors can be used three times. Thus, there must be two colors each used exactly twice. The probability of this is
$$\frac{\frac{1}{2}\binom{4}{2} \cdot 3 \cdot 2}{3^4} = \frac{2}{9},$$
since there are $\frac{1}{2}\binom{4}{2}$ ways to separate the four faces into two sets of two faces and $3 \cdot 2$ ways to choose two colors in order to color them. The desired probability is $1 - \frac{2}{9} = \frac{7}{9}$.

17. Let $(k, \sqrt{k})$ be the point on the graph of $y = \sqrt{x}$ that is closest to the point $(2017, 0)$. Find k.

Proposed by: Conway Xu

ANSWER: $\boxed{\frac{4033}{2}}$

SOLUTION: Consider a point $(x, \sqrt{x})$ so as to minimize $\sqrt{(x-2017)^2 + \left(\sqrt{x} - 0\right)^2} = \sqrt{x^2 - 4033x + 2017^2}$. Clearly this is minimized when $x = \frac{4033}{2}$.

18. Alice is going to place 2016 rooks on a 2016×2016 chessboard where both the rows and columns are labelled 1 to 2016; the rooks are placed so that no two rooks are in the same row or the same column. The value of a square is the sum of its row number and column number. The score of an arrangement of rooks is the sum of the values of all the occupied squares. Find the average score over all valid configurations.

Proposed by: Yiming Zheng

ANSWER: $\boxed{4066272}$

SOLUTION: Form a bijection between arrangements and permutations of $(1,2,3,\cdots,2016)$ as follows: for a permutation $(\sigma_1,\sigma_2,\sigma_3,\cdots,\sigma_{2016})$ place a rook in the ith row and σ_ith column. Now let S_{2016} be the set of permutations of $(1,2,3,\cdots,2016)$. We wish to compute

$$\begin{aligned}\frac{1}{2016!}\left[\sum_{\sigma\in S_{2016}}(i+\sigma_i)\right] &= \frac{1}{2016!}\left[\sum_{\sigma\in S_{2016}} i+\sum_{\sigma\in S_{2016}}\sigma_i\right]\\ &= \frac{1}{2016!}\left[2016!\cdot\tfrac{(2016)(2017)}{2}+2016!\cdot\tfrac{(2016)(2017)}{2}\right]\\ &= 2016\cdot 2017\\ &= 4066272.\end{aligned}$$

19. Let $f(n)$ be a function defined recursively across the natural numbers such that $f(1)=1$ and $f(n)=n^{f(n-1)}$. Find the sum of all positive divisors less than or equal to 15 of the number $f(7)-1$.

Proposed by: Yiming Zheng

ANSWER: $\boxed{88}$

SOLUTION: Clearly $1,2,3,4,6,8$ are divisors by taking $f(7)$ mod these numbers. 5 is also a divisor because $f(7)=7^{f(6)}\equiv 2^{f(6)} \pmod 5$, and $f(6)\equiv 0 \pmod 4$, so $f(7)=2^{f(6)}\equiv 1 \pmod 5$. By Chinese Reaminder Theorem, $10,12,15$ all work. It is also easy to see that $7,14$ are not divisors. Finally, one can check 11 does not work: $f(6)\equiv 0 \pmod 2$, $f(6)\equiv 1 \pmod 5 \implies f(6)\equiv 6 \pmod{10} \implies f(7)=7^{f(6)}\equiv 7^6 \pmod{11}\neq 1$. Also, it is easy to check 13 works: $f(6)\equiv 0 \pmod{12} \implies 7^{f(6)}\equiv 7^0=1 \pmod{11}$. So the requested sum is

$$1+2+3+4+5+6+8+9+10+12+13+15=88$$

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20. Find the number of ordered pairs of positive integers (m,n) that satisfy

$$\gcd(m,n)+\operatorname{lcm}(m,n)=2017.$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{8}$

SOLUTION: Note that $d=\gcd(m,n)$ divides both sides of the equation, hence either $d=2017$ or $d=1$. If $d=2017$, then

$$\gcd(m,n)+\operatorname{lcm}(m,n)>2017.$$

If $d=1$, we have $mn+1=2017 \implies mn=2016=2^5\cdot 3^2\cdot 7$. Since m and n are relatively prime, we just have to choose which prime factors from the set $\{2,3,5\}$ divide m. This can be done in 8 ways, so there are 8 ordered pairs.

21. Let $\triangle ABC$ be a triangle. Let M be the midpoint of AB and let P be the projection of A onto BC. If $AB=20$, and $BC=MC=17$, compute BP.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{100}{17}}$

SOLUTION: Let Q be the projection of C onto AB. Since $\triangle BCM$ is isosceles, we have $BQ=\frac{BM}{2}=\frac{AB}{4}=5$. Let N be the midpoint of BC. We have $BN=\frac{BC}{2}=\frac{17}{2}$. By power of a point at B with respect to the nine point circle of $\triangle ABC$, we have $BM\cdot BQ=BN\cdot BP \implies 10\cdot 5=\frac{17}{2}\cdot BP$, from which it follows that $BP=\frac{100}{17}$.

22. For positive integers n, define the odd parent function, denoted $\text{op}(n)$, to be the greatest positive odd divisor of n. For example, $\text{op}(4) = 1, \text{op}(5) = 5$, and $\text{op}(6) = 3$. Find

$$\sum_{i=1}^{256} \text{op}(i).$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{21846}$

SOLUTION: We have

$$\begin{aligned}
\sum_{i=1}^{256} \text{op}(i) &= 1 + \sum_{i=0}^{7} \sum_{j=1}^{2^{7-i}} \text{op}(2^i(2j-1)) \\
&= 1 + \sum_{i=0}^{7} \sum_{j=1}^{2^{7-i}} (2j-1) \\
&= 1 + \sum_{i=0}^{7} ((2^{7-i})(2^{7-i}+1) - 2^{7-i}) \\
&= 1 + \sum_{i=0}^{7} 4^{7-i} \\
&= 1 + \sum_{i=0}^{7} 4^{i} \\
&= 1 + \frac{4^8 - 1}{3} \\
&= 21846.
\end{aligned}$$

23. Suppose $\triangle ABC$ has sidelengths $AB = 20$ and $AC = 17$. Let X be a point inside $\triangle ABC$ such that $BX \perp CX$ and $AX \perp BC$. If $|BX^4 - CX^4| = 2017$, the compute the length of side BC.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{\sqrt{223887}}{111}}$

SOLUTION: Notice that

$$|BX^4 - CX^4| = |BX^2 - CX^2| \cdot |BX^2 + CX^2| = |AB^2 - AC^2| \cdot |BC^2| = 3 \cdot 37 \cdot BC^2 = 2017,$$

so

$$BC = \sqrt{\frac{2017}{111}} = \frac{\sqrt{223887}}{111}.$$

24. How many ways can some squares be colored black in a 6×6 grid of squares such that each row and each column contain exactly two colored squares? Rotations and reflections of the same coloring are considered distinct.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{67950}$

SOLUTION: Let A_n be the answer when 6 is replaced with n and call the squares (i, j) for $i, j \in \{1, 2, \cdots, n\}$. Assume $(1,1), (1,2), (2,1)$ are colored without loss of generality. We will multiply by $\binom{n}{2} \cdot (n-1)$ at the end. If $(2,2)$ is colored there are A_{n-2} ways to finish. Else the remaining $(n-1) \times (n-1)$ board can be filled by a valid coloring subject to the constraint that the lower left square is colored, and then we may remove this square. This happens in $\frac{2}{n-1}$ cases since columns can always be permuted. Thus,

$$A_n = \frac{n(n-1)^2}{2}\left(A_{n-2} + \frac{2 \cdot A_{n-1}}{n-1}\right).$$

We have

$$\begin{aligned} A_2 &= 1 \\ A_3 &= 6 \\ A_4 &= 90 \\ A_5 &= 2040, \end{aligned}$$

and it follows that $A_6 = 67950$.

25. Let $ABCD$ be a convex quadrilateral with $AB = BC = 2$, $AD = 4$, and $\angle ABC = 120^\circ$. Let M be the midpoint of BD. If $\angle AMC = 90^\circ$, find the length of segment CD.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{2\sqrt{7}}$

SOLUTION: Let N be the midpoint of AC and let P be the midpoint of CD. Notice that N is the midpoint of the hypotenuse of triangle $\triangle AMC$, so $NM = \frac{AC}{2} = \sqrt{3}$. We also have that $MP = \frac{BC}{2} = 1$ and $NP = \frac{AD}{2} = 2$. It follows that $\angle MPN = 60^\circ$. Since $MP \parallel BC$ and $NP \parallel AD$, the angle of intersection between lines AD and BC must also be 60°. Suppose AD and BC intersect at Q. Then $\angle AQB = 60^\circ$ and $\angle ABQ = 60^\circ$, so $\angle BAD = 120^\circ$. It follows that $\angle DAC = \angle BAD - \angle BAC = 120^\circ - 30^\circ = 90^\circ$. Thus, we have $CD = \sqrt{AC^2 + AD^2} = \sqrt{12+16} = 2\sqrt{7}$.