Theme Round

Lexington High School

April 9, 2016

Move-y

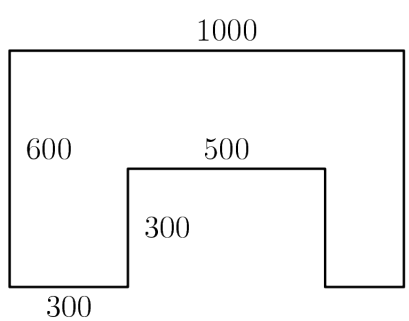
1. Memories all must have at least one out of five different possible colors, two of which are red and green. Furthermore, they each can have at most two distinct colors. If all possible colorings are equally likely, what is the probability that a memory is at least partly green given that it has no red?

2. Mater is confused and starts going around the track in the wrong direction. He can go around 7 times in an hour. Lightning and Chick start in the same place at Mater and at the same time, both going the correct direction. Lightning can go around 91 times per hour, while Chick can go around 84 times per hour. When Lightning passes Chick for the third time, how many times will he have passed Mater (if Lightning is passing Mater just as he passes Chick for the third time, count this as passing Mater)?

3. Geri plays chess against himself. White has a 5% chance of winning, Black has a 5% chance of winning, and there is a 90% chance of a draw. What is the expected number of games Geri will have to play against himself for one of the colors to win four times?

4. A male volcano is in the shape of a hollow cone with the point side up, but with everything above a height of 6 meters removed. The resulting shape has a bottom radius of 10 meters and a top radius of 7 meters, with a height of 6 meters. He sat above his bay, watching all the couples play. His lava grew and grew until he was half full of lava. Then, he erupted, lowering the height of the lava to 2 meters. What fraction of the lava remained in the volcano?

5. Pixar Prison, for Pixar villains, is shaped like a 600 foot by 1000 foot rectangle with a 300 foot by 500 foot rectangle removed from it, as shown below. The warden must separate it into three congruent polygonal sections for villains from The Incredibles, A Bug's Life, and Cars 2. What is the perimeter of each of these sections?



Func-y

- A set is a collection of elements. For example, the set $\{1,2,3,4\}$ is a set with elements $1,2,3$, and 4.
- If x is an element of a set S, we write $x \in S$.
- $[x, y)$ denotes the set of all real numbers greater than or equal to x and less than y.
- A function $f : A \to B$, where A and B are sets, is a function with domain A and codomain B. This means that the only elements which f takes as inputs are all of the elements of A, and the only elements which f outputs are (not necessarily all of) the elements of B. If the function takes an element $x \in A$ to an element $y \in B$, we write $f(x) = y$.
- A function $f : A \to B$ is called injective if for every element $y \in B$, there is at most one element $x \in A$ such that $f(x) = y$.
- A function $f : A \to B$ is called surjective if for every element $y \in B$, there is at least one element $x \in A$ such that $f(x) = y$.
- A function is called bijective if it is both injective and surjective.

6. How many functions $f : \{1,2,3,4\} \to \{1,2,3\}$ are surjective?

7. Let $R(x)$ be a function that takes a natural number as input and returns a rectangle. $R(1)$ is known to have integer side lengths. Let $p(x)$ be the perimeter of $R(x)$ and let $a(x)$ be the area of $R(x)$. Suppose that $p(x+5) = 6p(x)$ for all x in the domain of R and that $a(x+2) = 12a(x)$ for all $x > 6$ in the domain of R. For $x \le 6$, $a(x+1) = a(x)+2$. Suppose $p(16) = 1296$, and let the side lengths of $R(11)$ be a and b with $a \le b$. Find the ordered pair (a, b).

8. Consider the function $f : [0,1) \to [0,1)$ defined by $f(x) = 2x - \lfloor 2x \rfloor$, where $\lfloor 2x \rfloor$ is the greatest integer less than or equal to $2x$. Find the sum of all values of x such that $f^{17}(x) = x$.

9. A function $f : \{1,2,3,\cdots,2016\} \to \{1,2,3,\cdots,2016\}$ is called *good* if the function $g(n) = |f(n) - n|$ is injective. Furthermore, a good function f is called *excellent* if there exists another good function f' such that $f(n) - f'(n)$ is nonzero for exactly one value of n. Let N be the number of good functions that are not excellent. Find the remainder when N is divided by 1000.

10. Let $S = \{1,2,3,4,5,6\}$. Find the number of bijective functions $f : S \to S$ for which there exist exactly 6 bijective functions $g : S \to S$ such that $f(g(x)) = g(f(x))$ for all $x \in S$.

Torn-y

11. A single elimination tournament is held with 2016 participants. In each round, players pair up to play games with each other. There are no ties, and if there are an odd number of players remaining before a round then one person will get a bye for the round. Find the minimum number of rounds needed to determine a winner.

12. A round robin tournament is held with 2016 participants. Each round, after seeing the results from the previous round, the tournament organizer chooses two players to play a game with each other that will result in a win for one of the players and a loss for the other. The tournament organizer wants each person to have a different total number of wins at the end of k rounds. Find the minimum possible value of k for which this can always be guaranteed.

13. A round robin tournament is held with 2016 participants. Each player plays each other player once and no games result in ties. We say a pair of players A and B is a *dominant pair* if all other players either defeat A and B or are defeated by both A and B. Find the maximum number dominant pairs.

14. A ladder style tournament is held with 2016 participants. The players begin seeded $1, 2, \cdots 2016$. Each round, the lowest remaining seeded player plays the second lowest remaining seeded player, and the loser of the game gets eliminated from the tournament. After 2015 rounds, one player remains who wins the tournament. If each player has probability of $\frac{1}{2}$ to win any game, then the probability that the winner of the tournament began with an even seed can be expressed has $\frac{p}{q}$ for coprime positive integers p and q. Find the remainder when p is divided by 1000.

15. A round robin tournament is held with 2016 participants. Each player plays each other player once and exactly one game results in a tie. Let W be the sum of the squares of each team's win total and let L be the sum of the squares of each team's loss total. Find the maximum possible value of $W - L$.