Team Round

Lexington High School

April 9, 2016

[70] Potpourri

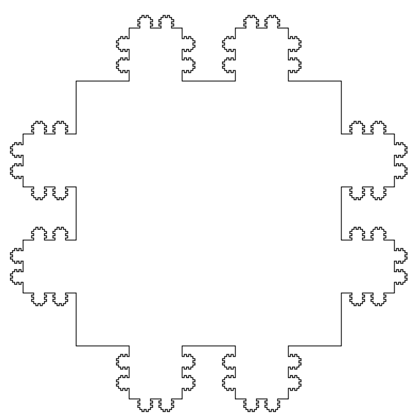
1. Let X, Y, Z be nonzero real numbers such that the quadratic function $Xt^2 - Yt + Z = 0$ has the unique root $t = Y$. Find X.

2. Let $ABCD$ be a kite with $AB = BC = 1$ and $CD = AD = \sqrt{2}$. Given that $BD = \sqrt{5}$, find AC.

3. Find the number of integers n such that $n - 2016$ divides $n^2 - 2016$. An integer a divides an integer b if there exists a unique integer k such that $ak = b$.

4. The points $A(-16, 256)$ and $B(20, 400)$ lie on the parabola $y = x^2$. There exists a point $C(a, a^2)$ on the parabola $y = x^2$ such that there exists a point D on the parabola $y = -x^2$ so that $ACBD$ is a parallelogram. Find a.

5. Figure F_0 is a unit square. To create figure F_1, divide each side of the square into equal fifths and add two new squares with sidelength $\frac{1}{5}$ to each side, with one of their sides on one of the sides of the larger square. To create figure F_{k+1} from F_k, repeat this same process for each open side of the smallest squares created in F_n. Let A_n be the area of F_n. Find $\lim_{n\to\infty} A_n$.



6. For a prime p, let S_p be the set of nonnegative integers n less than p for which there exists a nonnegative integer k such that $2016^k - n$ is divisible by p. Find the sum of all p for which p does not divide the sum of the elements of S_p.

7. Trapezoid $ABCD$ has $AB \parallel CD$ and $AD = AB = BC$. Unit circles γ and ω are inscribed in the trapezoid such that circle γ is tangent to CD, AB, and AD, and circle ω is tangent to CD, AB, and BC. If circles γ and ω are externally tangent to each other, find AB.

8. Let x, y, z be real numbers such that $(x+y)^2+(y+z)^2+(z+x)^2 = 1$. Over all triples (x, y, z), find the maximum possible value of $y-z$.

9. Triangle $\triangle ABC$ has sidelengths $AB = 13, BC = 14$, and $CA = 15$. Let P be a point on segment BC such that $\frac{BP}{CP} = 3$, and let I_1 and I_2 be the incenters of triangles $\triangle ABP$ and $\triangle ACP$. Suppose that the circumcircle of $\triangle I_1PI_2$ intersects segment AP for a second time at a point $X \neq P$. Find the length of segment AX.

10. [**BONUS**] For $1 \leq i \leq 9$, let A_i be the answer to problem i from this section. Let $(i_1, i_2, \cdots, i_9)$ be a permutation of $(1,2,\cdots,9)$ such that $A_{i_1} < A_{i_2} < \cdots < A_{i_9}$. For each i_j, put the number i_j in the box which is in the jth row from the top and the jth column from the left of the 9×9 grid in the bonus section of the answer sheet. Then, fill in the rest of the squares with digits $1,2,\cdots,9$ such that

    - each bolded 3×3 grid contains exactlty one of each digit from 1 to 9,
    - each row of the 9×9 grid contains exactly one of each digit from 1 to 9, and
    - each column of the 9×9 grid contains exactly one of each digit from 1 to 9.

    To receive credit for this problem you **do not** need to have correctly solved all of the earlier problems.

[130] Long Answer

Maggie is a budding artist who recently learned to paint, but she is constantly distracted by Evan's antics. For your solutions in this section, use the below definitions. Remember to justify all your answers fully, and to prove and clearly explain all of your arguments.

- A *set* is an empty or non-empty collection of objects, without repeats.
- A *element* of a set S is one of the objects contained in S. If $A_1, A_2, A_3, \ldots$ are these objects, we write $S = \{A_1, A_2, A_3, \ldots\}$.
- A *subset* of a set S is a set with only elements that are in S.
- The *power set* of a set S, also written as $\mathscr{P}(S)$, is the set consisting of all subsets of S (Yes – We can have a set of sets!).
- The *empty set*, also written as $\varnothing$, contains no elements and is thus a subset of every set.
- A *coloring* of a set S is a way to label the elements of S by either assigning each one a color or leaving it "plain".

Let $T = \{A, B, C, D, E\}$. Matt arrives early at the studio one morning, and decides to add a splash of color to T by making some, all, or none of the elements blue.

1. [**2**] How many ways can Matt color T?

2. [**2**] How many elements are in $\mathscr{P}(T)$?

Later in the morning, Matt finds a bucket of gold paint, and decides to color the elements of $\mathscr{P}(T)$.

3. [**3**] How many ways can Matt color $\mathscr{P}(T)$ by making all, some, or none of the elements gold?

Suppose Evan wants to color $\mathscr{P}(T)$ according to the following rule: If a set contains at least as many vowels as consonants, then it is gold.

4. [**3**] What fraction of the sets in $\mathscr{P}(T)$ are gold?

Let $S = \{1, 2, 3, \ldots, 2016\}$. Evan wants to color $\mathscr{P}(S)$ red according to the following rules:

a) For any two elements Evan picks in S, there is exactly one red subset containing both of them.

b) Whenever Evan picks a red subset R and an element s outside of R, he can find at least two red subsets X, Y that contain s and share no elements with R.

5. [**15**] What is the maximum possible number of red sets in Evans's coloring?

Now, Evan is coloring $\mathscr{P}(T)$, where $T = \{A, B, C, D, E\}$, according to the same rules.

6. [**20**] How many distinct colorings can Evan make?

Maggie finds some orange paint and decides to color according to some new rules. She is coloring $\mathscr{P}(T)$, where $T = \{A, B, C, D, E\}$, so that:

a) The empty set, $\varnothing$, is orange.

b) The set T is orange.

c) If sets X and Y are both orange, then $X \cup Y$ and $X \cap Y$ are both orange as well. $X \cup Y$ is the set of all elements in either X or Y, and $X \cap Y$ is the set of all elements in both X and Y.

7. [**5**] Give an example, or prove that there is no example, of a way that Maggie could color $\mathscr{P}(T)$ by coloring 7 sets orange.

8. [**20**] How many ways can Maggie color by these rules so that no one-element set is orange?

9. [**15**] Find the largest number of sets that Maggie could color orange without coloring every set orange, and construct an example of such a coloring. Prove that you have found the maximal case.

Maggie and Evan decide to play a game called **PIG** (Paint It Goldenrod) with the set $Y = \{1, 2, \ldots, 17\}$. Using a fresh bottle of CAP'N NATHAN'S YELLOW DYE ™, they take turns coloring elements of $\mathscr{P}(Y)$ yellow. A player wins the game if after any move the set satisfies his or her painting rules. The game ends after there is a winner, although if both players win at the same time, they tie. Evan will use the rules used for Problems 5 and 6. Maggie will use the rules used for Problems 7, 8, and 9.

10. [**5**] Can **PIG** end in a tie?

11. [**15**] If both players play optimally, who wins? Does it matter who goes first? Give winning strategies, for both cases if they are different, or prove that there is no consistent winning strategy.

Let $Z = \{A, B, C, D\}$. Define the operation *symmetric difference* on two sets A and B, denoted $\ominus$, by $A \ominus B = A \cup B \setminus A \cap B$. In other words, $A \ominus B$ is the set of elements in A or B, but not both. Peter wants to color the elements of $\mathscr{P}(Z)$ green according to the following rule: If A and B are two not necessarily distinct green elements, then $A \ominus B$ is also a green green.

12. [**15**] Prove that, if he colors a nonzero number of subsets, then the number of subsets that Peter colors must be a power of 2.

13. [**10**] Find the number of ways that Peter can color all, some, or none of the elements of $\mathscr{P}(Z)$ green.