2015 LMT Solutions

LHS Math Club

March 28, 2015

1 Individual Round

1. 1, obviously.

2. The ratio of the side lengths of these 2 squares is $1/\sqrt{2}$, so the ratio of their areas is $1/2$. Notice that no information on what the side lenghts actually are is needed.

3. The previous year is 11110001111, which is 1967.

4. The common ratio in the geometric progression is $101/10$, so the next term is $104060401 \cdot 101/10 = 1051010050.1$.

5. Be careful about the variable names! Since $\sqrt{A/\pi} = r = \pi \cdot A^2$, we have that $A = 1/\pi$, so $C = 2A = 2/\pi$.

6. By manipulating the given equations, we see that $T - O = 9$ and $I + T + N = E + T + N = 10$, so $I = E$. But then $N + E = 9/2$, so $T = 11/2$ and $O = -7/2$.

7. Suppose N has a digits. Then $10(N - 6 \cdot 10^{a-1}) + 6 = N/4$, so $N = (6 \cdot 10^a - 6)/39 = (2 \cdot 10^a - 3)/13$, meaning that $2 \cdot 10^a - 2$ must be a multiple of 13. The smallest a for which this holds is $a = 6$, giving us our answer of 615384.

8. Let $x = 2014$. Then the expression inside the cube root is equal to $x(x-1)(x+1) + x = x^3$, so the desired answer is $x = 2014$.

9. A number is divisible by 11 if the sum of the odd-numbered digits is congruent to the sum of the even-numbered digits modulo 11 (so the sum of the 1st and 3rd digits and the sum of the 2nd and 4th digits are congruent). Because the digits that we are dealing with are 1,2,3, and 4, the two sums must be equal, because it is not possible to get a sum that is 11 more than the other. Thus 1,4 must be in the odd-numbered digits or the even-numbered digits, and 2,3 in the other slots. There are 2 ways to pick odd or even for 1,4, then 2 ways to arrange 1,4, and then 2 ways to arrange 2,3, so there are a total of $2^3 = 8$ such numbers.

10. Since there is replacement, the answer is the same as that for a 4-card deck: in this case, the probability of 4 distinct suits is $4!/4^4 = 3/32$.

11. Let the side length of LMT be 1. Then $HL = \sqrt{3}/2 + 1/2$, and $AL = \sqrt{3}/2 - 1/2$. So the desired ratio is $(\sqrt{3}+1)/(\sqrt{3}-1) = 2 + \sqrt{3}$.

12. It is easy to show that if a "2" is ever carried in this equation, then we can not get an answer. Then since a "1" must always be carried where necessary and all of the digits are distinct, $S+H = 10$, $H+G = I+L = 9$, $2H - 9 = C$, and $C + 1 = O$. Notice that since $2H - 9 = C$, we must have that $H \geq 5$. By trying different possibilities for H, we obtain that $H = 7$, $\{L, I\} = \{8, 1\}$, $G = 2$, $C = 5$, $O = 6$, and $S = 3$. Then the remaining variables which are not in the equation, $\{M, A, T\} = \{0, 4, 9\}$. Even though all of the variables are not completely determined, the desired ordered pair is always the same and is equal to $(20, 15)$.

13. There are 3 cases to consider: (1) none of the first 3 people have the Green: the desired probability in this case is $3/5 \cdot 2/4 \cdot 1/3 = 1/10$. (2) The third person has the Green: the desired probability in this case is $3/5 \cdot 2/4 \cdot 1/3 \cdot 1/2 = 1/20$. (3) Either the first or second person has the Green: the desired probability is $2 \cdot 1/5 \cdot 3/4 \cdot 2/3 \cdot 1/2 = 1/10$. Overall, the desired probability that the 3rd person gets the Green is $1/10 + 1/10 + 1/20 = 1/4$.

14. The 8 possibilities are: $21 * 2 * 2 * 2 * 12, 12 * 2 * 2 * 2 * 21, 2 * 21 * 2 * 12 * 2, 2 * 12 * 2 * 21 * 2, 42 * 2 * 24, 24 * 2 * 42, 21 * 8 * 12$, and $12 * 8 * 21$.

15. We are given a standard linear recurrence, so we can write $S_n = a \cdot 2^n + b \cdot (-1)^n$ for appropriate a, b. Using that $S_1 = 3, S_2 = 4$, we find that $a = 7/6, b = -2/3$. We now can sum the two geometric series given by $\sum_{n=1}^{\infty} a \cdot (2/3)^n$ and $\sum_{n=1}^{\infty} b \cdot (-1/3)^n$ to achieve an answer of $5/2$.

16. 7028491635; it is easy to verify that this is correct. To come up with this, we might note that there are relatively few powers of two and multiples of 7 with at most two digits, so we could list them out and play around with them until we find the correct ordering.

17. Let n be the desired fraction. Then $n = (8+n)/(3+n)$, so $n = 2$.

18. We put the triangles in the circle so that there are two triangles per semicircle: one is isosceles, and the other is right, with a hypotenuse which is the diameter of the circle (both triangles have all 3 vertices on the circle). Let this diameter be AB, let the center of the circle be O, let the third vertex of the right triangle be C, and let the isosceles triangle be CBD. We want to find the area of each of these triangles, given that the areas are equal. Letting M be the midpoint of BC, we note that $MD = 2MO$ since BCD and ABC have equal area. But then since $OD = 1$, we have that $OM = 1/3$, so $AC = 2OM = 2/3$, and the area of ABC is $1/2 \cdot 2/3 \cdot \sqrt{8/9} = 4\sqrt{2}/9$.

19. The answer is simply the sum of the divisors of 120, divided by 120. This is $(1+2+4+8)\cdot(1+3)\cdot(1+5)/120 = 3$.

20. By conditioning on the central element of the arithmetic progression, we obtain that the number of progressions is $2\cdot(1+2+\cdots+1999) = 3998000$.

2 Theme Round

2.1 Songs

1. If W is Meghan's weight and b is the base, then we have that $W = 4b^2+5b+1 = 4b^2+4b+7$. Solving for b and then W, we obtain that $W = 175$.

2. Taylor's "song rate" is 7 songs per 4 years. Then song 1989 comes out in 3136.

3. Sir Mix-a-lot is B; he likes big butts. D does not like big butts, C does not like big butts, and A likes big butts.

4. Notice that the temperature of Bruno Mars at time t is $2^{t+1}+2^t-t-2$. This can be proven by considering sums of the form $\sum_{i=0}^{n} i\cdot 2^i$. For relatively large t, this is approximately equal to 2^{t+1}, so we want $2^{t+1} \geq 1,000,000$, and the smallest t that satisfies this is $t = 19$.

5. Since j is a multiple of 3 and a perfect square, it must be a multiple of 9. Thus the sum of the digits of j is a multiple of 9 as well, meaning that z is a multiple of 27. It is then easy to see that $(j,z) = (2916, 54)$.

2.2 Physics

1. The buildings must be 0 feet apart in order for a 20-foot rope to droop down 10 feet.

2. We want the geometriz series $\sum_{i=0}^{\infty} 4\cdot(4/5)^i = 20$.

3. The answer is $3/2^2 = 3/4$.

4. The probability that Albert gets all of the particles correct is $1/\binom{12}{6}$, and the probability that he gets exactly 10 of the particles correct is $6\cdot 6/\binom{12}{6}$, giving an answer of 37/924.

5. The planets which take 77 and 33 years to orbit will first be collinear with the sun after $11\cdot 21/8$ years; this is the number of years it takes the faster planet to gain "half an orbit" on the smaller planet. The same is true for the planets with orbits 33 and 21 years, so all three planets will in fact be collinear after 231/8 years.

2.3 Frisbee

1. Only kids at positions which are multiples of $\gcd(100, 2015) = 5$ will touch the frisbee. Hence the number who do not touch it is $2015-2015/5 = 1612$.

2. First of all, the LMT of each player must be 9; if the LMTs of all players were not equl, then the LMT of some player would be greater than the LMT of both of its neighbors, which is impossible. Thus the probability that the frisbee will make it all the way back to Ivan after 10 throws is $1/10^{10}$. (These guys have trouble throwing frisbees.)

3. Let the center of the circle be O, and denote points on the circle by the first letter of the name of the person standing there. Notice that angle COZ must be 60 degrees in order for angle ZOH, which is 3 times COZ to satisfy $ZH = 2CZ$. But then $CH = \sqrt{3}$.

4. The only diagonals of a dodecagon which can be written as $\sqrt{i}$, where $i \in N$, have lengths $1, \sqrt{2}, \sqrt{3}, 2$; the squares of these lengths are $1, 2, 3, 4$. If there are N throws and S is the sum of the squares of their distances, then we want $(N-1)+S = 11 \Rightarrow N+S = 12$. If $N = 3$, then we can have 3 throws of lengths $\sqrt{3}$. If $N = 4$, we can have 4 throws of length $\sqrt{2}$. If $N = 5$, we can have throws of lengths $1, 1, 1, \sqrt{2}, \sqrt{2}$. And if $N = 6$, we can have 6 throws of length 1. It is easy to check that no other N-value works, so the sum of all possible N is 18.

5. If we let x be the number of "17-throws", then we want $17x+11(187-x) \equiv 0 \pmod{187}$, meaning that $x \equiv 0 \pmod{187}$. However, this is clearly impossible, so there are 0 ways to pass the frisbee.