Team Round Solutions

LMT Spring 2023

May 20, 2023

1. **[10]** Given the following system of equations:

$$\begin{cases} RI+G+SP &= 50 \\ RI+T+M &= 63 \\ G+T+SP &= 25 \\ SP+M &= 13 \\ M+RI &= 48 \\ N &= 1 \end{cases}$$

Find the value of L that makes $LMT+SPRING = 2023$ true.

Proposed by

Solution. $\boxed{\frac{341}{40}}$

$$(RI+T+M)-(M+RI) = T = 63-48 = 15$$
$$(G+T+SP)-T = G+SP = 25-15 = 10$$
$$(RI+G+SP)-(G+SP) = RI = 50-10 = 40$$
$$(M+RI)-RI = M = 48-40 = 8$$
$$(SP+M)-M = SP = 13-8 = 5$$
$$(G+T+SP)-T-SP = 25-15-5 = 5$$

Thus, $RI = 40$, $G = 5$, $SP = 5$, $T = 15$, $M = 8$.

$$L\cdot 8\cdot 15+5\cdot 40\cdot 5 = 2023$$
$$120L+1000 = 2023$$
$$120L = 1023$$
$$L = \frac{1023}{120} = \boxed{\frac{341}{40}}$$

□

2. **[11]** How many integers of the form n^{2023-n} are perfect squares, where n is a positive integer between 1 and 2023 inclusive?

Proposed by Muztaba Syed

Solution. $\boxed{1034}$

If n is odd, then $2023-n$ is even, so it is guaranteed to be a perfect square. There are 1012 values like this. If n is even then the exponent won't do anything to make it a perfect square, meaning n itself has to be a perfect square. Out of the perfect squares less than 2023 ($1^2, 2^2, 3^2 \dots 44^2$) there are 22 even values. This means our answer is $1012+22 = \boxed{1034}$ □

3. **[12]** Beter Pai wants to tell you his fastest 40-line clear time in Tetris, but since he does not want Qep to realize she is better at Tetris than he is, he does not tell you the time directly. Instead, he gives you the following requirements, given that the correct time is t seconds:

   - $t < 100$.
   - t is prime.
   - $t-1$ has 5 proper factors.
   - all prime factors of $t+1$ are single digits.
   - $t-2$ is a multiple of 3.
   - $t+2$ has 2 factors.

   Find t.

   Proposed by Samuel Wang

   Solution. $\boxed{29}$

   Solution: requirement 2 states that t is prime, thus $t = 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97$. Thus, $t+1 = 3,4,6,8,12,14,18,20,24,30,32,38,42,44,48,54,60,62,68,72,74,80,84,90,98$. Of these, only $t+1 = 3,4,6,8,12,14,18,20,24$ satisfy requirement 4. Thus, $t-1 = 1,2,4,6,10,12,16,18,22,28,30,40,46,52,58,70,78,82,88,96$. Of these, only $t-1 = 12,18,28,52$ work. Requirement 5 narrows this down to $t = 29,53$ and requirement 6 narrows this down to $\boxed{t = 29}$ □

4. **[13]** There exists a certain right triangle with the smallest area in the 2D coordinate plane such that all of its vertices have integer coordinates but none of its sides are parallel to the x- or y-axis. Additionally, all of its sides have distinct, integer lengths. What is the area of this triangle?

   Proposed by Peter Bai

   Solution. $\boxed{150}$

   Solution:

   Since the legs are not parallel to the x- or y-axis, they themselves must be the hypotenuse of a smaller right triangle with legs parallel to the x- and y-axis. As a result, both legs of the original triangle must be the hypotenuse of a Pythagorean triple.

   Since the original triangle itself is a right triangle itself with integer sides, it itself must also be a Pythagorean triple whose legs are themselves hypotenuses of Pythagorean triples!

   To minimize its area, we try to use the smallest triples possible to build up this triangle - that is, we will only use multiples of 3-4-5 triangles to obtain the legs as well as the final triangle itself. Since the legs of the original triangle, at minimum, must be in a ratio of 3:4, the legs must be 3*5 and 4*5 respectively. This is because both legs must be from a multiple of the same Pythagorean triple, 3-4-5 (due to the fact that they must be perpendicular). Our final diagram looks like the attached image.

   By Shoelace Bash, the area of the black triangle is $\boxed{150}$. □

5. **[14]** How many ways are there to place the integers from 1 to 8 on the vertices of a regular octagon such that the sum of the numbers on any 4 vertices forming a rectangle is even? Rotations and reflections of the same arrangement are considered distinct.

   Proposed by Muztaba Syed

   Solution. $\boxed{12672}$

   A rectangle consists of 2 pairs of diametrically opposite points. There are 4 such pairs, and the even condition implies that they are either all odd, or all even. Case 1: They are all odd. This means every even number is paired with an odd. First we fix the even numbers: $8\cdot6\cdot4\cdot2$. This gives the odds $4\cdot3\cdot2\cdot1$ options. This gives $8\cdot6\cdot4\cdot2\cdot24$. Case 2: They are all even. This means odds are paired with odds and evens are paired with evens. There are $3\cdot3 = 9$ ways to assign these pairs. There are 4! ways to arrange the pairs on the circle, times 2^4 for which element of the pair goes where. This gives a total of $9\cdot16\cdot24$. Now computing the total:

   $$24\cdot(8\cdot6\cdot4\cdot2+9\cdot16) = 24\cdot(384+144) = 24\cdot528 = \boxed{12672}$$

   □

6. **[16]** Find the least positive integer m such that $105 \mid 9^{(p^2)} - 29^p + m$ for all prime numbers $p > 3$.

   Proposed by Samuel Wang

   Solution. $\boxed{20}$

   We do mod 3, mod 5, and mod 7 mod 3 gives $3|m+1$ mod 5 gives $5|m$ mod 7 gives $7|m+1$ 20 mod 105 works answer is thus $\boxed{20}$ □

7. **[18]** Jerry writes down all binary strings of length 10 without any two consecutive 1s. How many 1s does Jerry write?

   Proposed by Evin Liang

   Solution. $\boxed{420}$

   $a_0 = 0, a_1 = 1, a_{n+2} = a_{n+1} + a_n + F_{n+1}$ □

8. **[20]** Let x, y, and z be positive reals that satisfy the system

$$\begin{cases} x^2 + xy + y^2 &= 10 \\ x^2 + xz + z^2 &= 20 \\ y^2 + yz + z^2 &= 30 \end{cases}$$

   Find $xy + yz + xz$.

   Proposed by Derek Zhao

   Solution. $\boxed{\dfrac{20\sqrt{6}}{3}}$

   x, y, z are positive reals so they can be represented as lengths. Draw $AD = x, BD = y, CD = z$. Also, $\angle ADB = \angle BDC = \angle CDA = 120°$. We claim $AB = \sqrt{10}, BC = \sqrt{20}, CA = \sqrt{30}$ because the equations are the law of cosines equation with $\theta = 120°$. Also, by the sin area formula $[ABC] = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}(xy + yz + xz)$. However, $[ABC] = \frac{1}{2} \cdot \sqrt{10} \cdot \sqrt{20}$ because $\angle ABC = 90°$. Solving the equation, we get $xy + yz + xz = \frac{20\sqrt{6}}{3}$. □

9. **[22]** In $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let E and F be the feet of the altitudes from B onto CA, and C onto AB, respectively. A line ℓ is parallel to EF and tangent to the circumcircle of ABC on minor arc BC. Let X and Y be the intersections of ℓ with AB and AC respectively. Find XY.

   Proposed by Muztaba Syed

   Solution. $\boxed{\dfrac{455}{24}}$

   First using the fact that $\triangle AEF \sim \triangle ABC$:

$$\angle ABC = \angle AEF = \angle AYX$$

   Which means that $\triangle ABC \sim \triangle AYX$. Let the point of tangency be Z, and the circumcenter O. Z is the intersection of the perpendicular from O to EF with the circumcircle of ABC. But this line contains A (because circumcenter orthocenter properties). Thus Z is the point diametrically opposite A. To find the ratios of the triangles, note that AZ is an altitude of $\triangle AYX$, which corresponds to the altitude from A to BC. AZ is a diameter, so it has length $2R = \frac{65}{4}$. The other altitude is 12, so our answer is:

$$BC \cdot \frac{65}{4} \div 12 = \frac{14 \cdot 65}{48} = \frac{455}{24} \implies \boxed{455024}$$

   □

10. **[24]** The sequence $a_0, a_1, a_2, \ldots$ is defined such that $a_0 = 2 + \sqrt{3}$, $a_1 = \sqrt{5 - 2\sqrt{5}}$, and

$$a_n a_{n-1} a_{n-2} - a_n + a_{n-1} + a_{n-2} = 0.$$

   Find the least positive integer n such that $a_n = 1$.

   Proposed by Jerry Xu

Solution. $\boxed{5}$

Note that $a_0 = \tan 75°$ and $a_1 = \tan 36°$. This motivates us to rearrange the given equation to get that

$$\begin{aligned} a_{n-1} + a_{n-2} &= a_n - a_n a_{n-1} a_{n-2} \\ a_{n-1} + a_{n-2} &= a_n(1 - a_{n-1}a_{n-2}) \\ a_n &= \frac{a_{n-1} + a_{n-2}}{1 - a_{n-1}a_{n-2}}. \end{aligned}$$

which is simply the tangent addition formula. Thus, we get that if $a_k = \tan\theta_k$,

$$\theta_k = \theta_{k-1} + \theta_{k-2}.$$

We know that $\theta_0 = 75°$ and $\theta_1 = 36°$, and we want $\theta_n \equiv 45°, 225° \pmod{2\pi}$. Listing out the first few θ_n, we get that

$$\begin{aligned} \theta_2 &= 75 + 36 \equiv 111°, \\ \theta_3 &= 36 + 111 \equiv 147°, \\ \theta_4 &= 111 + 147 \equiv 258°, \\ \theta_5 &= 147 + 258 \equiv 45°, \end{aligned}$$

so our answer is $\boxed{5}$. □