Speed Round Solutions

LMT Spring 2023

May 20, 2023

1. **[6]** Evaluate $(2-0)^2 \cdot 3 + \frac{20}{2+3}$.

   Proposed by Samuel Wang

   Solution. $\boxed{16}$

   Solution: By simple computation, this is $12+4=\boxed{16}$ □

2. **[6]** Let $x = 11 \cdot 99$ and $y = 9 \cdot 101$. Find the sum of the digits of $x \cdot y$.

   Proposed by Jacob Xu

   Solution. $\boxed{36}$

   Solution: $xy = 99 \cdot 9999 = (100-1)(10000-1) = 1000000 - 10000 - 100 + 1 = 998901$, thus the answer is $\boxed{36}$. □

3. **[6]** A rectangle is cut into two pieces. The ratio between the areas of the two pieces is $3:1$ and the positive difference between those areas is 20. What's the area of the rectangle?

   Proposed by Corey Zhao

   Solution. $\boxed{40}$

   $3x - 1x = 20$, so $x = 10$, so the area is 40 □

4. **[6]** Edgeworth is scared of elevators. He is currently on floor 50 of a building, and he wants to go down to floor 1. Edgeworth can go down at most 4 floors each time he uses the elevator. What's the minimum number of times he needs to use the elevator to get to floor 1?

   Proposed by Jacob Xu

   Solution. $\boxed{13}$

   Using the elevator 12 times gets you down to floor 2, so you need to use it once more giving us an answer of 13 □

5. **[6]** There are 20 people at a party. Fifteen of those people are normal and 5 are crazy. A normal person will shake hands once with every other normal person, while a crazy person will shake hands twice with every other crazy person. How many total handshakes occur at the party?

   Proposed by Jacob Xu

   Solution. $\boxed{125}$

   15C2 + 2*5C2 = 125 □

6. **[6]** Wam and Sang are chewing gum. Gum comes in packages, each package consisting of 14 sticks of gum. Wam eats 6 packs and 9 individual sticks of gum. Sang wants to eat twice as much gum as Wam. How many packs of gum must Sang buy?

   Proposed by Samuel Wang

Solution. $\boxed{14}$

Solution: Twice of 6 packs and 9 sticks of gum is 12 packs and 18 sticks of gum. Since more than 14 individual sticks of gum, is one pack, he must have bought. $12+1+1=\boxed{14}$ □

7. **[6]** At Lakeside Health School (LHS), 40% of students are male and 60% of the students are female. If half of the students at the school take biology, and the same number of male and female students take biology, to the nearest percent, what percent of female students take biology?

   Proposed by

   Solution. $\boxed{42}$

   $25/60 = 41.66\%$ □

8. **[6]** Evin is bringing diluted raspberry iced tea to the annual Lexington Math Team party. He has a cup with 10 mL of iced tea and a 2000 mL cup of water with 10% raspberry iced tea. If he fills up the cup with 20 more mL of 10% raspberry iced tea water, what percent of the solution will be iced tea?

   Proposed by Samuel Wang

   Solution. $\boxed{40}$

   He has 10 mL iced tea + (18 mL water + 2 mL iced tea -> $12/30$ is iced tea, which is 40%. □

9. **[6]** Tree 1 starts at height 220m and grows continuously at 3m per year. Tree 2 starts at height 20m and grows at 5m during the first year, 7m per during the second year, 9m during the third year, and in general $(3+2n)$m in the nth year. After which year is Tree 2 taller than Tree 1?

   Proposed by Calvin Garces

   Solution. $\boxed{14}$

   So during the first year, Tree 1 is initially 200m taller than Tree 2. At the start of the 2nd year, the difference is $200-2=198$m. 3rd year, $198-4=194$. In general Tree 1 - Tree 2 at year n is $200-(n+1)(n)$. This means that Tree 1 - Tree 2 will be less than 0 when n is 14, so the answer is 14. □

10. **[6]** Leo and Chris are playing a game in which Chris flips a coin. The coin lands on heads with probability $\frac{499}{999}$, tails with probability $\frac{499}{999}$, and it lands on its side with probability $\frac{1}{999}$. For each flip of the coin, Leo agrees to give Chris 4 dollars if it lands on heads, nothing if it lands on tails, and 2 dollars if it lands on its side. What's the expected value of the number of dollars Chris gets after flipping the coin 17 times?

    Proposed by Aidan Duncan

    Solution. $\boxed{34}$

    Clearly, Chris's expected gain per flip is 2 dollars. So, the answer is $2\cdot 17=\boxed{34}$. □

11. **[6]** Ephram has a pile of balls, which he tries to divide into piles. If he divides the balls into piles of 7, there are 5 balls that don't get divided into any pile. If he divides the balls into piles of 11, there are 9 balls that aren't in any pile. If he divides the balls into piles of 13, there are 11 balls that aren't in any pile. What is the minimum number of balls Ephram has?

    Proposed by

    Solution. $\boxed{999}$

    -2 mod 7, 11, 13 □

12. **[6]** Let $\triangle ABC$ be a triangle such that $AB=3$, $BC=4$, and $CA=5$. Let F be the midpoint of AB. Let E be the point on AC such that $EF \parallel BC$. Let CF and BE intersect at D. Find AD.

    Proposed by Derek Zhao

Solution. $\boxed{2\sqrt{13}/3}$

E is the midpoint of AC because $EF \parallel BC$. Therefore, D is the centroid, so AD is two-thirds the length of the median from A. Let the median from A be AM. $BM = 2$ and $AB = 3$, so $AM = \sqrt{13}$. Therefore, $AD = \frac{2\sqrt{13}}{3}$. □

13. **[6]** Compute the sum of all even positive integers $n \le 1000$ such that:

$$\text{lcm}(1,2,3,\dots(n-1)) \neq \text{lcm}(1,2,3,\dots n).$$

Proposed by Muztaba Syed

Solution. $\boxed{1022}$

As we increase n by 1, observe that this lcm value changes only when n is a power of a prime. Otherwise each of its prime factors have already appeared and it contributes nothing to the lcm. The only even prime powers are powers of 2 (this won't include 1). $2+4+8\dots+512 = 1024-1-1 = \boxed{1022}$. □

14. **[6]** Find the sum of all palindromes with 6 digits in binary, including those written with leading zeroes.

Proposed by Boyan Litchev

Solution. $\boxed{252}$

There are $2^3 = 8$ such numbers. Each of their digits will be $\frac{0+1}{2} = \frac{1}{2}$ on average (for each palindrome, we can invert all the digits to get another palindrome), meaning that on average each number will be $\frac{111111_2}{2} = \frac{63}{2}$. So, the sum is $8 \cdot 632 = 252$ □

15. **[6]** What is the side length of the smallest square that can entirely contain 3 non-overlapping unit circles?

Proposed by Peter Bai

Solution. $\boxed{2+\dfrac{\sqrt{2}+\sqrt{6}}{2}}$

Answer: $2+\frac{\sqrt{2}+\sqrt{6}}{2}$ is trivial with knowledge of 15-75-90 triangles. However, I don't see a good way to do it without that □

16. **[6]** Find the sum of the digits in the base 7 representation of 6250000. Express your answer in base 10.

Proposed by Aidan Duncan

Solution. $\boxed{16}$

Converting to base 7, $6250000 = 50^4$ is $(101)^4 = 104060401 \implies \boxed{16}$. □

17. **[6]** A number n is called *sus* if n^4 is one more than a multiple of 59. Compute the largest sus number less than 2023.

Proposed by Evin Liang

Solution. $\boxed{2007}$

Since 59 is -1 mod 4, n is sus if and only if it is congruent to ± 1 mod 59. The largest integer less than 2023 congruent to -1 mod 59 is 20025 and the largest integer less than 2023 congruent to 1 mod 59 is 2007, so the largest sus number less than 2023 is 2007. □

18. **[6]** Michael chooses real numbers a and b independently and randomly from $(0,1)$. Given that a and b differ by at most $\frac{1}{4}$, what is the probability a and b are both greater than $\frac{1}{2}$?

Proposed by Michael Han

Solution. $\boxed{3/7}$

Graphing the region where a and b differ by at most $\frac{1}{4}$ we find it has area $\frac{7}{16}$. Then we calculate the area within this region that also satisfies $a > \frac{1}{2}$ and $b > \frac{1}{2}$, this turns out to have area $\frac{3}{16}$. Thus, our final probability is $\frac{\frac{3}{16}}{\frac{7}{16}} = \boxed{\frac{3}{7}}$. □

19. **[6]** In quadrilateral $ABCD$, $AB = 7$ and $DA = 5$, $BC = CD$, $\angle BAD = 135^\circ$ and $\angle BCD = 45^\circ$. Find the area of $ABCD$.

Proposed by Samuel Wang

Solution. $\boxed{36+36\sqrt(2)}$

We can rotate quadrilateral ABCD around C 8 times, forming an octagon centered at C with side length AD+AB=12. By symmetry, this octagon has 8 times the area of ABCD. Now, the area of this octagon is $288+288\sqrt{2}$, thus the answer is $\boxed{36+36\sqrt{2}}$ □

20. **[6]** Find the value of

$$\sum_{i|210} \sum_{j|i} \left\lfloor \frac{i+1}{j} \right\rfloor$$

Proposed by Muztaba Syed

Solution. $\boxed{1276}$

So the key (and dumb) observation is that most of the time $\left\lfloor \frac{i+1}{j} \right\rfloor$ is equal to $\frac{i}{j}$ unless $j = 1$. Also $\frac{i}{j}$ is always an integer. So we have:

$$\sum_{i|210} \sum_{j|i} \left\lfloor \frac{i+1}{j} \right\rfloor = \sum_{i|210} \left(\sum_{j|i} \frac{i}{j} + 1 \right) = d(210) + \sum_{i|210} \sigma(i)$$

So then we factor the sum:

$$d(210) + (1+(1+2)) \cdot (1+(1+3)) \cdot (1+(1+5)) \cdot (1+(1+7)) = 16+4\cdot 5\cdot 7\cdot 9 = \boxed{1276}$$

□

21. **[6]** Let a_n be the number of words of length n with letters $\{A, B, C, D\}$ that contain an odd number of As. Evaluate a_6.

Proposed by Samuel Wang

Solution. $\boxed{2016}$

Solution: There are 2 cases when trying to go from a sequence with length $n-1$ to one with length n: Case 1: adding something other than A. There are 3 other letters to add and a_{n-1} sequences to add them to, giving $3(a_{n-1})$ Case 2: adding A. This requires the previous sequence to have an even number of As, which can happen in $4^{n-1} - a_{n-1}$ ways. Thus, $a_n = 4^{n-1} + 2a_{n-1}$ We thus compute: $a_1 = 1$, $a_2 = 6$, $a_3 = 28$, $a_4 = 120$, $a_5 = 496$, $a_6 = \boxed{2016}$ □

22. **[6]** Detective Hooa is investigating a case where a criminal stole someone's pizza. There are 69 people involved in the case, among whom one is the criminal and another is a witness of the crime. Every day, Hooa is allowed to invite any of the people involved in the case to his rather large house for questioning. If on some given day, the witness is present and the criminal is not, the witness will reveal who the criminal is. What is the minimum number of days of questioning required such that Hooa is guaranteed to learn who the criminal is?

Proposed by Samuel Wang

Solution. $\boxed{8}$

Solution: It is necessary and sufficient to have everyone in the case showing up on a different set of days, none of which is a subset of any other. If person A goes on all days person B does, if person A is the criminal and B the witness, this is bad. Otherwise, the criminal and witness have different sets of days, meaning there is a day where the witness shows up but the criminal does not. Thus, it is possible for the investigation to last $\boxed{8}$ days where everyone shows up for a different set of 4 days. There are 70 such sets, thus this is possible. 7 days is impossible without having 2 sets A and B where $A \subseteq B$ (proof needed), thus the answer is 8. □

23. **[6]** Find
$$\sum_{n=2}^{\infty} \frac{2n+10}{n^3+4n^2+n-6}.$$

Proposed by Peter Bai

Solution. $\boxed{\frac{19}{12}}$

Testing some small numbers, we can see that $(x-1)$ is a factor of the denominator. Factoring, we get $\sum_{n=1}^{\infty} \frac{2n+10}{(n-1)(n+2)(n+3)}$.

Now, we can use partial fractions. Let a, b, c be real numbers such that $\frac{2n+10}{(n-1)(n+2)(n+3)} = \frac{a}{n-1} + \frac{b}{n+2} + \frac{c}{n+3}$ for all real n except $1, -2$, and -3.

Multiplying both sides by $(n-1)(n+2)(n+3)$ gives $2n+10 = a(n+2)(n+3) + b(n-1)(n+3) + c(n-1)(n+2)$.

Because partial fractions (you can substitute values for n that would make the original equation undefined here), we can plug in $n = 1, -2, -3$ into the equation, which rapidly simplifies to yield $a = 1, b = -2$, and $c = 1$ respectively.

Going back to the sum, after evaluating a few terms under our partial fraction decomposition, we can see that all terms with denominators greater than or equal to 5 cancel out. Our answer is thus $1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{19}{12}}$. □

24. **[6]** Let $\triangle ABC$ be a triangle with circumcircle ω such that $AB = 1$, $\angle B = 75°$, and $BC = \sqrt{2}$. Let lines ℓ_1 and ℓ_2 be tangent to ω at A and C respectively. Let D be the intersection of ℓ_1 and ℓ_2. Find $\angle ABD$ (in degrees).

Proposed by Derek Zhao

Solution. $\boxed{30}$

Line AD is the symmedian line, which means that the line is the reflection of the median over the angle bisector. Let M be the midpoint of AC. $\angle ABD = \angle CBM$ because AD is the symmedian line. Using the ratio lemma, $\frac{\sin(\angle ABM)}{\sin(\angle CBM)} = \sqrt{2}$. Let $\angle CBM = x$.
$$\frac{\sin(75-x)}{\sin(x)} = \frac{\frac{\sqrt{6}+\sqrt{2}}{4}\cos(x) - \frac{\sqrt{6}-\sqrt{2}}{4}\sin(x)}{\sin(x)} = \frac{\frac{\sqrt{6}+\sqrt{2}}{4}\cos(x)}{\sin(x)} - \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Therefore,
$$\frac{\frac{\sqrt{6}+\sqrt{2}}{4}\cos(x)}{\sin(x)} - \frac{\sqrt{6}-\sqrt{2}}{4} = \sqrt{2}.$$
$$\frac{\cos(x)}{\sin(x)} = \frac{\sqrt{6}+3\sqrt{2}}{4} \cdot \frac{4}{\sqrt{6}+\sqrt{2}} = \sqrt{3}.$$

Therefore, $x = 30°$ so the answer is 30. □

25. **[6]** Find the sum of the prime factors of $14^6 + 27$.

Proposed by Evin Liang

Solution. $\boxed{597}$

$a^6 + 27b^6 = (a^2+3b^2)(a^2+3ab+3b^2)(a^2-3ab+b^2)$ so $14^6 + 27 = 157 \cdot 199 \cdot 241$ which are all prime. □