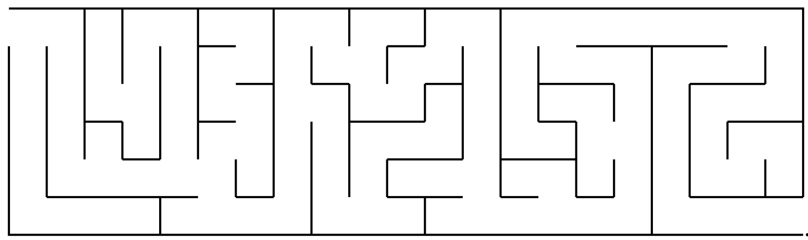
LMT Spring 2023 Guts Round Solutions- Part 1

Team Name: ______________

______________ 1. **[4]** Solve the maze



Proposed by Andrew Zhao

Solution. $\boxed{Writeonmaze}$

Solving the maze gives a shape that appears to spell out LMT. □

______________ 2. **[4]** Billiam can write a problem in 30 minutes, Jerry can write a problem in 10 minutes, and Evin can write a problem in 20 minutes. Billiam begins writing problems alone at 3:00 PM until Jerry joins him at 4:00 PM, and Evin joins both of them at 4:30 PM. Given that they write problems until the end of math team at 5:00 PM, how many full problems have they written in total?

Proposed by Jerry Xu

Solution. $\boxed{11}$

Billiam writes 4 problems, Jerry writes six, and Evin writes one. The total is $\boxed{11}$. □

______________ 3. **[4]** How many pairs of positive integers (n,k) are there such that $\binom{n}{k} = 6$?

Proposed by Corey Zhao

Solution. $\boxed{3}$

(4,2), (6,1), (6,5) □

LMT Spring 2023 Guts Round Solutions- Part 2

Team Name: ______________

______________ 4. **[5]** Find the sum of all integers $b > 1$ such that the expression of 143 in base b has an even number of digits and all digits are the same.

Proposed by Muztaba Syed

Solution. $\boxed{154}$

Because the number of digits is even, by grouping consecutive digits we see that $b+1$ divides 143. The only factors of 143 are 1, 11, 13, and 143. So we have $b = 142$, 12, or 10. Checking these shows that 12 and 142 work, for an answer of $\boxed{154}$. □

______________ 5. **[5]** Ini thinks that $a\#b = a^2 - b$ and $a\&b = b^2 - a$, while Mimi thinks that $a\#b = b^2 - a$ and $a\&b = a^2 - b$. Both Ini and Mimi try to evaluate $6\&(3\#4)$, each using what they think the operations & and # mean. What is the positive difference between their answers?

Proposed by Jacob Xu

Solution. $\boxed{4}$

For Ini, $3 \# 4 = 3^2 - 4 = 5$ and $6 \& 5 = 5^2 - 6 = 19$. For Mimi, $3 \# 4 = 4^2 - 3 = 13$ and $6 \& 13 = 36 - 13 = 23$. So the answer is $23 - 19 = \boxed{4}$ □

______ 6. **[5]** A unit square sheet of paper lies on an infinite grid of unit squares. What is the maximum number of grid squares that the sheet of paper can partially cover at once? A grid square is partially covered if the area of the grid square under the sheet of paper is nonzero - i.e., lying on the edge only does not count.

Proposed by Peter Bai

Solution. $\boxed{6}$

A construction for 6 is easy - simply position the sheet of paper with its center at coordinates (0, 0.5) and rotate it. To prove that 7 is impossible, we only need to check a square rotated 45 degrees and slightly offset from the origin. Simple geometry shows that this cannot contain both bottom corners of the center square and the top middle, so it will not overlap with 7 squares. □

LMT Spring 2023 Guts Round Solutions- Part 3

Team Name:

7. **[6]** Maya wants to buy lots of burgers. A burger without toppings costs \$4, and every added topping increases the price by 50 cents. There are 5 different toppings for Maya to choose from, and she can put any combination of toppings on each burger. How much would it cost for Maya to buy 1 burger for each distinct set of toppings? Assume that the order in which the toppings are stacked onto the burger does not matter.

Proposed by Jacob Xu

Solution. $\boxed{168}$

There are 32 different burgers, with an average of 2.5 toppings, the price being 5.25. Multiplying this gives 168. □

8. **[6]** Consider square $ABCD$ and right triangle PQR in the plane. Given that both shapes have area 1, $PQ = QR$, $PA = RB$, and P, A, B and R are collinear, find the area of the region inside both square $ABCD$ and $\triangle PQR$, given that it is not 0.

Proposed by Corey Zhao

Solution. $\boxed{3/4}$

As the area of PQR is 1 and it is a right triangle, we know that its dimensions must be $\sqrt{2}$, $\sqrt{2}$, and 2, where $PR = 2$. Thus, $PA = RB = \frac{1}{2}$. Then, we can see that the region of PQR outside of $ABCD$ is two isosceles right triangles, both with legs of length $\frac{1}{2}$. Hence the area of those two outside triangles will be $2*(\frac{1}{2})^2 \cdot \frac{1}{2} = \frac{1}{4}$. Then the region inside both $ABCD$ and PQR will be $1 - \frac{1}{4} = \frac{3}{4}$, as the remainder of PQR is inside $ABCD$, which we can see by the height from Q to PR being exactly 1. □

9. **[6]** Find the sum of all n such that n is a 3-digit perfect square that has the same tens digit as $\sqrt{n}$, but that has a different ones digit than $\sqrt{n}$.

Proposed by Samuel Wang

Solution. $\boxed{1258}$

$23^2 + 27^2 = 529 + 729 = \boxed{1258}$. □

LMT Spring 2023 Guts Round Solutions- Part 4

Team Name:

10. **[7]** Jeremy writes the string:

LMTLMTLMTLMTLMTLMT

on a whiteboard ("LMT" written 6 times). Find the number of ways to underline 3 letters such that from left to right the underlined letters spell LMT.

Proposed by Muztaba Syed

Solution. $\boxed{56}$

We just need to decide which chunk "LMT" each character comes from. This can be done with a grid argument (which gives an answer of $\binom{8}{3} = 56$) or just:

$$\binom{6}{3} + 6 \cdot 5 + 6 = \boxed{56}$$

The $\binom{6}{3}$ comes from them being from different chunks, $6 \cdot 5$ is using 2 distinct chunks, and 6 is all from the same chunk. □

11. **[7]** Compute the remainder when 12^{2022} is divided by 1331.

Proposed by Evin Liang

Solution. $\boxed{1310}$

By the binomial theorem $12^{2022} = 1 + 2022 \cdot 11 + \binom{2022}{2} \cdot 121$ plus something that is divisible by 1331. Also, $\binom{2022}{2} \cdot 121$ mod 1331 only depends on $\binom{2022}{2}$ mod 11 so $\binom{2022}{2} \cdot 121$ is 363 mod 1331, $2022 \cdot 11$ is 946 mod 1331, so 12^{2022} is 1310 mod 1331. □

12. **[7]** What is the greatest integer that cannot be expressed as the sum of 5s, 23s, and 29s?

Proposed by Evin Liang

Solution. $\boxed{47}$

First compute the smallest number in each residue class mod 5 that can be expressed as the sum of 5s, 23s and 29s. Since it is the smallest in the residue class, we do not use any 5s. Then the possible smallest numbers in each residue class are $23a + 29b$ where a and b are from 0 to 4. Looking at the values of $23a$ the value of 0 mod 5 is at most 0, 1 mod 5 is at most 46, 2 mod 5 is at most 92, 3 mod 5 is at most 23, and 4 mod 5 is at most 69. Each value is therefore at most 92, so we do not need to consider values of $23a + 29b$ that are greater than 92. The other values are 29, 52, 75, 58, 81, and 87. Then we see that for 0 mod 5 the smallest is 0, for 1 mod 5 the smallest is 46, for 2 mod 5 the smallest is 52, for 3 mod 5 the smallest is 23, and for 4 mod 5 the smallest is 29. Thus, the largest number in each residue class that cannot be represented are 41, 47, 18, and 24 and the largest among them is 47. □

LMT Spring 2023 Guts Round Solutions- Part 5

Team Name:

13. **[9]** Square $ABCD$ has point E on side BC, and point F on side CD, such that $\angle EAF = 45°$. Let $BE = 3$, and $DF = 4$. Find the length of FE.

Proposed by Ben Yin

Solution. $\boxed{7}$

Consider rotating ABE 90 degrees clockwise about A. Then, $E'F = 7$ and $E'AF$ and EAF are congruent from SAS. Thus, $EF = 7$. □

14. **[9]** Find the sum of all positive integers k such that

1. k is the power of some prime.
2. k can be written as 5654_b for some $b > 6$.

Proposed by Samuel Wang

Solution. $\boxed{2048}$

Solution: This is $5b^3 + 6b^2 + 5b + 4 = (b+1)(5b^2 + b + 4)$. By Euclidean Algorithm, the gcd between the 2 factors is a factor of 8, thus both $b+1$ and $5b^2 + b + 4$ are powers of 2. Trying $b = 7$ gives $b + 1 = 8$, $5b^2 + b + 4 = 256$. This gives $5654_7 = \boxed{2048}$ □

15. **[9]** If $\sqrt[3]{x} + \sqrt[3]{y} = 2$ and $x + y = 20$, compute $\max(x, y)$

Proposed by Evin Liang

Solution. $\boxed{10 + 6\sqrt{(3)}}$

Cubing the first equation gives $x + y + 3\sqrt[3]{xy}(\sqrt[3]{x} + \sqrt[3]{y}) = 8$, so $xy = -8$. Therefore x and y are the solutions to $t^2 - 20t - 8 = 0$, which are $10 \pm 6\sqrt{3}$. The larger solution is $10 + 6\sqrt{3}$. □

16. **[11]** Triangle ABC with $AB < AC$ is inscribed in a circle. Point D lies on the circle and point E lies on side AC such that $ABDE$ is a rhombus. Given that $CD = 4$ and $CE = 3$, compute AD^2.

Proposed by Muztaba Syed

Solution. $\boxed{44}$

AD is the angle bisector of $\angle BAC$, so $CD = BC = 4$. Because $ABDE$ is a rhombus, AB is also equal to these CD and BC. Also $AC = AE + EC = 4 + 3 = 7$. Now because $CD = AB$, we see that $ABCD$ is an isosceles trapezoid, meaning $AD = BC$. Now by Ptolemy's Theorem:

$$AC \cdot BD + AB \cdot CD = AD \cdot BC \implies 7 \cdot 4 + 4 \cdot 4 = AD^2 = \boxed{44}$$

□

17. **[11]** Wam and Sang are walking on the coordinate plane. Both start at the origin. Sang walks to the right at a constant rate of 1 m/s. At any positive time t (in seconds), Wam walks with a speed of 1 m/s with a direction of t radians clockwise of the positive x-axis. Evaluate the square of the distance between Wam and Sang in meters after exactly 5π seconds.

Proposed by Samuel Wang

Solution. $\boxed{25\pi^2 + 4}$

Notice that Wam walks in a circle centered at (0, 1) with radius 1with period 2π seconds, thus after the 5π seconds he will reach the point (0, 2). Sang would end up at $(5\pi, 0)$, thus the square of the distance between them is $\boxed{25\pi^2 + 4}$ □

18. **[11]** Mawile is playing a game against Salamance. Every turn, Mawile chooses one of two moves: Sucker Punch or Iron Head, and Salamance chooses one of two moves: Dragon Dance or Earthquake. Mawile wins if the moves used are Sucker Punch and Earthquake, or Iron Head and Dragon Dance. Salamance wins if the moves used are Iron Head and Earthquake. If the moves used are Sucker Punch and Dragon Dance, nothing happens and a new turn begins. Mawile can only use Sucker Punch up to 8 times. All other moves can be used indefinitely. Assuming both Mawile and Salamance play optimally, find the probability that Mawile wins.

Proposed by Stephanie Wan

Solution. $\boxed{\frac{8}{9}}$

The game ends after 9 turns. During these 9 turns, Mawile has to click Iron Head once, and Salamence wins if and only if it guesses the correct turn (it can commit to clicking Earthquake exactly once) □

19. **[13]** Ephram is attempting to organize what rounds everyone is doing for the NEAML competition. There are 4 rounds, of which everyone must attend exactly 2. Additionally, of the 6 people on the team (Ephram, Wam, Billiam, Hacooba, Matata, and Derke), exactly 3 must attend every round. In how many different ways can Ephram organize the teams like this?

Proposed by Samuel Wang

Solution. $\boxed{1860}$

Genfunc solution: This is simply the coefficient of $a^3b^3c^3d^3$ in $(ab+ac+ad+bc+bd+cd)^6$, which is $\boxed{1860}$. Now come on will anyone actually use this solution? It sucks.

Combo solution: There are $\binom{6}{3} = 20$ ways to choose who does round 1. Of those people, here are the cases: Case 1: all do the same round. There are 3 ways to determine what the other round is. This entirely determines what rounds the other 3 do, thus the answer here is 3×20 Case 2: 2 do a round, 1 does another. There are $3 \times 2 = 6$ ways of choosing what the other rounds are and 3 ways to choose who does each round. WLOG, say 2 do round 2 and 1 does round 3. Thus, of the remaining 3 people, all do round 4 and 1 is chosen for round 2 with the others doing round 3. This gives a result of 54×20 Case 3: all do different rounds. There are 6 ways of assigning these rounds, then of the other 3, 2 do each round. WLOG, say these 3 are Wam, Derke, and Billiam. There are 3 different sets of rounds these 3 may do, and thus $3! = 6$ arrangements of them. The answer in this case is thus 36×20

The answer is thus $93 \times 20 = \boxed{1860}$ □

20. **[13]** For some 4th degree polynomial $f(x)$, the following is true:

- $f(-1) = 1$.
- $f(0) = 2$.
- $f(1) = 4$.
- $f(-2) = f(2) = f(3)$.

Find $f(4)$.

Proposed by Peter Bai

Solution. $\boxed{6}$

We can begin by making a difference table. We have:

x	$f(x)$	$\Delta f(x)$	$\Delta\Delta f(x)$	$\Delta\Delta\Delta f(x)$
-2				
-1	1			
0	2	1		
1	4	2	1	
2				
3				

We clearly do not have enough information to find the polynomial from the first three lines of information alone. Let us denote $\Delta\Delta\Delta f(x)$ for $x = 2, 1$ as a, b respectively. We can continue filling out the table in terms of a and b to get:

x	$f(x)$	$\Delta f(x)$	$\Delta\Delta f(x)$	$\Delta\Delta\Delta f(x)$
-2	$1-b$			
-1	1	b		
0	2	1	$1-b$	
1	4	2	1	b
2	$a+7$	$a+3$	$a+1$	a
3	$5a-b+11$	$4a-b+4$	$3a-b+1$	$2a-b$

We now have 2 equations in a and b:

$f(-2) = f(2) \Rightarrow 1-b = a+7$ $f(-2) = f(3) \Rightarrow 1-b = 5a-b+11$

Solving, we get that $a = -2$ and $b = -4$. After adding one more row to the difference table, we can see that $f(4) = 15a - 5b + 16$ and thus $f(4) = 15(-2) - 5(-4) + 16 = -30 + 20 + 16 = \boxed{6}$. □

21. **[13]** Find the minimum value of the expression $\sqrt{5x^2 - 16x + 16} + \sqrt{5x^2 - 18x + 29}$ over all real x.

Proposed by Peter Bai

Solution. $\boxed{\sqrt(29)}$

We can split this expression up into its constituent square roots. Looking at the first one, $\sqrt{5x^2-16x+16}$, we can see that the $-16x+16$ terms look awfully like the terms in the expansion of $(2x-4)^2$. Substituting this into the root, we can see that $\sqrt{5x^2-16x+16} = \sqrt{x^2+(2x-4)^2}$. Trying a similar tactic with the second root, we can see that it also breaks down nicely into a sum of squares: $\sqrt{5x^2-18x+29} = \sqrt{(x-5)^2+(2x-2)^2}$.

Taking a good look at this new representation of the expression, we can now see that each root looks quite like a distance formula between two points. In fact, if we substitute $y=2x$, that is exactly what this expression becomes.

$\sqrt{x^2+(2x-4)^2}+\sqrt{(x-5)^2+(2x-2)^2} = \sqrt{x^2+(y-4)^2}+\sqrt{(x-5)^2+(y-2)^2}$

So, out of all of the points lying on the line $y=2x$, we are looking for one that minimizes the sum of the distances between itself and the points (0,4) and (5,2). This happens when said point lies on the line that passes through (0,4) and (5,2), meaning that we simply need to find the intersection of the two lines $y=2x$ and, using point-slope form, $y-4=-\frac{2}{5}(x-0)$ or $y=-\frac{2}{5}x+4$. Solving, we get $x=\frac{5}{3}$.

Plugging this into the original expression, we get $\sqrt{\frac{29}{9}}+\sqrt{\frac{116}{9}} = \frac{\sqrt{29}}{3}+\frac{2\sqrt{29}}{3} = \boxed{\sqrt{29}}$. □

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______________ 22. **[15]** Let O and I be the circumcenter and incenter, respectively, of $\triangle ABC$ with $AB = 15$, $BC = 17$, and $CA = 16$. Let $X \neq A$ be the intersection of line AI and the circumcircle of $\triangle ABC$. Find the area of $\triangle IOX$.

Proposed by William Hua

Solution. $\boxed{85\sqrt{(21)}/168}$

Drop the altitude from I to OX.

To find the circumradius of the triangle, use the formula $R = \frac{abc}{4[ABC]}$, where a, b, and c are the side lengths. $[ABC]$ can be calculated with Heron's formula, so $[ABC] = 24\sqrt{21}$. Therefore, $R = \frac{15\cdot16\cdot17}{24\sqrt{21}} = \frac{85}{2\sqrt{21}}$.

Note that $BX = CX$ because AI bisects arc BC. Therefore, $OX \perp BC$.

Easy but coordinate way to get the altitude from I to OX:

If we let $B = (0,0)$ and $C = (17,0)$, then to get the x-coordinate of A, drop the altitude from A to BC, and let the foot be Y. Then, $BY = AB\cos B$, and $\cos B = \frac{225+289-256}{2*15*17} = \frac{258}{510} = \frac{43}{85}$, so $BY = \frac{129}{17}$. According to the coordinate formula of an incenter, the x-coordinate is $\frac{ax_A+bx_B+cx_C}{a+b+c}$, where x_A, x_B, and x_C are the x-coordinates of A, B, and C, respectively, and a, b, and c are BC, CA, and AB, respectively. Therefore, $\frac{ax_A+bx_B+cx_C}{a+b+c} = \frac{17\cdot\frac{129}{17}+16\cdot0+15\cdot17}{17+16+15} = \frac{129+255}{48} = 8$.

The x-coordinate of O is $\frac{17}{2}$, so the length of the altitude from I to OX is $\frac{17}{2} - 8 = \frac{1}{2}$. Then, $[IOX] = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{85}{2\sqrt{21}} = \frac{85}{8\sqrt{21}} = \frac{85\sqrt{21}}{168}$. □

______________ 23. **[15]** Find the sum of all integers x such that there exist integers y and z such that

$$x^2 + y^2 = 3(2016^z) + 77.$$

Proposed by Samuel Wang

Solution. $\boxed{208}$

2016 is a multiple of 7, thus $z < 2$. $z = 0$ and $z = 1$ both work. $z = 0$ gives $x = 4, 8$ and $z = 1$ gives $x = 35, 70, 14, 77$. This sum is thus $\boxed{208}$. □

______________ 24. **[15]** Evaluate

$$\left\lfloor \sum_{i=1}^{2022} \frac{1}{\sqrt{i}} \right\rfloor = \left\lfloor \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{2022}} \right\rfloor.$$

Proposed by Jerry Xu

Solution. $\boxed{88}$

Note that

$$\frac{2}{\sqrt{x}+\sqrt{x+1}} < \frac{1}{\sqrt{x}} < \frac{2}{\sqrt{x-1}+\sqrt{x}}.$$

Additionally, note that

$$\frac{2}{\sqrt{x}+\sqrt{x+1}} = \frac{2\left(\sqrt{x+1}-\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x+1}-\sqrt{x}\right)}$$

$$= \frac{2\left(\sqrt{x+1}-\sqrt{x}\right)}{\sqrt{x+1}^2-\sqrt{x}^2}$$

$$= 2\left(\sqrt{x+1}-\sqrt{x}\right).$$

Note that

$$\begin{aligned}\sum_{i=1}^{2022} 2\left(\sqrt{i+1}-\sqrt{i}\right) &= 2\left(\sqrt{2}-\sqrt{1}\right)+2\left(\sqrt{3}-\sqrt{2}\right)\cdots+2\left(\sqrt{2023}-\sqrt{2022}\right)\\ &= 2\left(\sqrt{2023}-\sqrt{1}\right)\end{aligned}$$

Since all the terms telescope except the first and last ones. Therefore,

$$\sum_{i=1}^{2022}\frac{1}{\sqrt{i}} > \sum_{i=1}^{2022} 2\left(\sqrt{i+1}-\sqrt{i}\right) = 2\left(\sqrt{2023}-\sqrt{1}\right).$$

Similarly,

$$\begin{aligned}\frac{1}{\sqrt{1}}+\sum_{i=2}^{2022} 2\left(\sqrt{i-1}-\sqrt{i}\right) &= 2\left(\sqrt{2}-\sqrt{1}\right)+2\left(\sqrt{3}-\sqrt{2}\right)\cdots+2\left(\sqrt{2022}-\sqrt{2021}\right)\\ &= \frac{1}{\sqrt{1}}+2\left(\sqrt{2022}-\sqrt{1}\right)\end{aligned}$$

(Note that we take the summation from $i = 2$ to 2022 since we cannot divide by 0), so therefore

$$\sum_{i=1}^{2022}\frac{1}{\sqrt{i}} < \sum_{i=2}^{2022} 2\left(\sqrt{i-1}-\sqrt{i}\right) = 2\left(\sqrt{2022}-\sqrt{1}\right)+1.$$

From here, we can simply note that $45^2 = 2025$ and $44^2 = 1936$, so 45 is the closest integer value to $\sqrt{2022}$ and $\sqrt{2023}$. Thus, the upper bound is approximately $2(45-1)+1 = 89$ and the lower bound is approximately $2(45-1) = 88$. Our answer is $\boxed{88}$. □

LMT Spring 2023 Guts Round Solutions- Part 9

Team Name:

______ 25. **[10]** Either:

1. Submit −2 as your answer and you'll be rewarded with two points OR

2. Estimate the number of teams that choose the first option. If your answer is within 1 of the correct answer, you'll be rewarded with three points, and if you are correct, you'll receive ten points

Proposed by Aidan Duncan

Solution. $\boxed{}$

□

______ 26. **[10]** Jeff is playing a turn-based game that starts with a positive integer n.

Each turn, if the current number is n, Jeff must choose one of the following:

1. The number becomes the nearest perfect square to n

2. The number becomes $n-a$, where a is the largest digit in n

Let $S(k)$ be the least number of turns Jeff needs to get from the starting number k to 0. Estimate

$$\sum_{k=1}^{2023} S(k).$$

If your estimation is E and the actual answer is A, you will receive $\max\left(\left\lfloor 10-\left|\frac{E-A}{6000}\right|\right\rfloor, 0\right)$ points.

Proposed by Aidan Duncan

Solution. $\boxed{232847}$

□

__________ 27. **[10]** Estimate the smallest positive integer n such that if N is the area of the n-sided regular polygon with circumradius 100, $10000\pi - N < 1$ is true.

If your estimation is E and the actual answer is A, you will receive $\max\left(\left\lfloor 10 - \left|10 \cdot \log_3\left(\frac{E}{A}\right)\right|\right\rfloor, 0\right)$ points.

Proposed by Aidan Duncan

Solution. $\boxed{455}$

□

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