Accuracy Round Solutions

LMT Spring 2023

May 20, 2023

1. **[6]** Andrew writes down all of the prime numbers less than 50. How many times does he write the digit 2?

   Proposed by Aidan Duncan

   Solution. $\boxed{3}$

   Primes can't end with 2, so $2, 23, 29$ are the only primes □

2. **[8]** Evaluate $2023^2 - 2022^2 + 2021^2 - 2020^2$.

   Proposed by Benjamin Yin

   Solution. $\boxed{8086}$

   Arithmetic □

3. **[8]** Phoenix is counting positive integers starting from 1. When he counts a perfect square greater than 1, he restarts at 1, skipping that square the next time. For example, the first 10 numbers Phoenix counts are $1, 2, 3, 4, 1, 2, 3, 5, 6, 7 \ldots$. How many numbers will Phoenix have counted after counting 100 for the first time?

   Proposed by Jacob Xu

   Solution. $\boxed{348}$

   $4 + 9 - 1 + 16 - 2 + 25 - 3 + 36 - 4 + 49 - 5 + 64 - 6 + 81 - 7 + 100 - 8 = 348$ □

4. **[10]** Rectangle $ABCD$ has side lengths $AB = 3$ and $BC = 7$. Let E be a point on BC, and let F be the intersection of DE and AC. Given that $[CDF] = 4$, find $\frac{DF}{FE}$.

   Proposed by Jerry Xu

   Solution. $\boxed{\dfrac{13}{8}}$

   We proceed by coordbash. Take A as the origin and $B = (3, 0)$. We thus have that $C = (3, 7)$ and $D = (0, 7)$. We can express point E as $(3, k)$. Line DE has equation $\dfrac{x(k-7)}{3} + 7$ and line AC has equation $y = \frac{7x}{3}$. Their intersection is $\left(\frac{21}{14-k}, \frac{49}{14-k}\right)$. Assume the foot of F with CD is G: we know that $[CDF] = \dfrac{CD \cdot FG}{2}$. $CD = 3$, so we have that $4 = \dfrac{3 \cdot FG}{2} \longrightarrow FG = \frac{8}{3}$. We also know that $FG = 7 - \dfrac{49}{14-k}$, so $\dfrac{8}{3} = 7 - \dfrac{49}{14-k}$. Multiplying both sides by $(14-k)$ and solving, we get that $k = \dfrac{35}{13}$. Thus, $E = \left(3, \dfrac{35}{13}\right)$ and $F = \left(\dfrac{13}{7}, \dfrac{13}{3}\right)$. We solve for DF and DE using pythag to get that $\dfrac{DF}{DE} = \boxed{\dfrac{13}{8}}$. □

5. **[10]** Let

$$N = \sum_{i=0}^{512} i \binom{512}{i}.$$

   What is the greatest integer a such that 2^a is a divisor of N?

   Proposed by Boyan Litchev

Solution. $\boxed{520}$

$N=\sum_{i=0}^{512} i\binom{512}{i} = \frac{1}{2}\left(\sum_{i=0}^{512} i\binom{512}{i}+\sum_{i=0}^{512} i\binom{512}{512-i}\right) = \frac{1}{2}\left(\sum_{i=0}^{512} i\binom{512}{i}+\sum_{i=0}^{512}(512-i)\binom{512}{i}\right) = \frac{1}{2}\sum_{i=0}^{512} 512\binom{512}{i} = 256\cdot 2^{512} = 2^{520}$. So, the largest possible a is 520. $\square$

6. **[10]** Aidan, Boyan, Calvin, Derek, Ephram, and Fanalex are all chamber musicians at a concert. In each performance, any combination of musicians of them can perform for all the others to watch. What is the minimum number of performances necessary to ensure that each musician watches every other musician play?

   Proposed by Samuel Wang

   Solution. $\boxed{4}$

   The answer is $\boxed{4}$. To achieve this, the first performance should consist of Aidan, Boyan, and Calvin performing, the second should be Calvin, Derek, and Ephram performing, the third should be Ephram, Fanalex, and Aidan performing, and the last should be Boyan, Derek, and Fanalex performing. It is easy to note that this works. We will now prove that 3 and lower do not work. Note that there are 30 pairs of musicians, counting order. Every performance, there are at most 9 pairs of musicians such that one performs for the other, thus in 3 or less performances at most 27 of these pairs can be fulfilled, which renders the problem impossible. Thus, as 4 is the minimum and is attainable, we are done. $\square$

7. **[12]** In $\triangle ABC$, $AB=13$, $BC=14$, and $CA=15$. Let D be a point on BC such that $BD=6$. Let E be a point on CA such that $CE=6$. Finally, let F be a point on AB such that $AF=6$. Find the area of $\triangle DEF$.

   Proposed by Samuel Wang

   Solution. $\boxed{\dfrac{288}{13}}$

   Solution: The area of ABC is 84. The area of AFE is $84\cdot\frac{6}{13}\cdot\frac{3}{5}=84\cdot\frac{18}{65}$, the area of BFD is $84\cdot\frac{7}{13}\cdot\frac{3}{7}=84\cdot\frac{3}{13}$, and the area of ECD is $84\cdot\frac{4}{7}\cdot\frac{2}{5}=84\cdot\frac{8}{35}$. Thus, the area of DEF is $84(1-\frac{18}{65}-\frac{3}{13}-\frac{8}{35})=84(\frac{455-126-105-104}{455})=84(\frac{335}{455})=84(\frac{67}{91})=84\cdot\frac{97}{13}=\frac{8148}{13}$, thus $A+B=\boxed{8161}$. $\square$

8. **[12]** Ephram is taking his final exams. He has 7 exams and his school holds finals over 3 days. For a certain arrangement of finals, let f be the maximum number of finals Ephram takes on any given day. Find the expected value of f.

   Proposed by Samuel Wang

   Solution. $\boxed{53/12}$

   Solution: f can be 3, 4, 5, 6, or 7 Ignoring order, there are 36 ways to arrange the exams. f=3: Ephram has 3 finals on one day. On the other 2 days, he either has 2 exams on both or 3 on one day and 1 on the other, both of which lead to 3 arrangements, for a total of 6. f=4: Ephram has 4 finals on one day. The distribution of finals is thus 4, 3, 0 or 4, 2, 1, each having 6 arrangements, for 12 total. f=5: the distribution is either 5, 2, 0 or 5, 1, 1, for 6 and 3 arrangements respectively, or 9 total. f=6: the distribution is 6, 1, 0, for 6 arrangements. f=7: the distribution is 7, 0, 0, for 3 arrangements.

   expected value is thus $\frac{3*3+4*12+5*9+6*6+7*3}{36}=\frac{53}{12}$ $\square$

9. **[12]** Evin's calculator is broken and can only perform 3 operations: Operation 1: Given a number x, output $2x$. Operation 2: Given a number x, output $4x+1$. Operation 3: Given a number x, output $8x+3$. After initially given the number 0, how many numbers at most 128 can he make?

   Proposed by Samuel Wang

   Solution. $\boxed{82}$

   Solution: Let a_k be the number of positive numbers at most 2^k that are attainable. We notice that $a_k=a_{k-1}+a_{k-2}+a_{k-3}$. Computing a few small values, we get $a_0=1$, $a_1=2$, $a_2=4$, $a_3=7$, $a_4=13$, $a_5=24$, $a_6=44$, $a_7=81$. Adding on 0 as a possibility gives $\boxed{82}$. $\square$

10. **[12]** Positive integers a, b, and c satisfy $a^2+b^2=c^3-1$ where $c \le 40$. Find the sum of all distinct possible values of c.

Proposed by William Hua

Solution. $\boxed{78}$

According to the sum of two squares theorem, a number can be expressed as a sum of squares if and only if there are no primes that are 3 mod 4 that has an odd exponent in the number's prime factorization. Note that one of the squares can be 0, and the number would be a perfect square.

If a number is 3 mod 4, this would mean that there is at least one prime that is 3 mod 4 that has an odd exponent in the number's prime factorization. Therefore, if c is even, then $c^3-1 \equiv 3 \bmod 4$, so c must be odd.

If $c = 1 \bmod 4$, then $a^2+b^2=c^3-1=(c-1)(c^2+c+1)$. $c^2+c+1 \equiv 1^2+1+1=3 \bmod 4$. Therefore, c^2+c+1 has at least one prime that is 3 mod 4 that has an odd exponent in its prime factorization. Let one of these primes be p.

If a prime q is 3 mod 4 and it divides a sum of squares u^2+v^2, then it must divide both u and v. Proof: if u isn't 0 mod q, then neither is v. Then, $u^2 \equiv -v^2 \bmod q$. Raise both sides to the $\frac{q-1}{2}$th power to get $1 \equiv u^{q-1} \equiv (-1)^{\frac{q-1}{2}} \cdot v^{q-1} \equiv -1 \bmod q$, a contradiction. Therefore, $q|u$ and $q|v$.

Let $a_k = \frac{a}{p^k}$, $b_k = \frac{b}{p^k}$, and $d_k = \frac{c^2+c+1}{p^{2k}}$. Therefore, $a_0^2+b_0^2=(c-1)(d_0)$. Dividing both sides by p^{2k} yields $a_k^2+b_k^2=(c-1)(d_k)$. Note that as long as d_k is an integer, it will always be divisible by p, because $v_p(d_k)$ will always be odd.

Let n be the largest positive integer such that d_n is an integer, so $v_p(d_n)=1$. Thus, $a_n^2+b_n^2=(c-1)(d_n)$. However, $p|(c-1)(d_n)=a_n^2+b_n^2$, so $p|a_n$ and $p|b_n$, and so $p^2|a_n^2+b_n^2$. Therefore, $p|c-1$ so $c \equiv 1 \bmod p$. However, $c^2+c+1 \equiv 1^2+1+1 \equiv 3 \equiv 0 \bmod p$, so p has to be 3. As a result, $c \equiv 1 \bmod 4$ and $c \equiv 1 \bmod 3$, so $c \equiv 1 \bmod 12$. Possible values of c are 1, 13, 25, and 37.

$1^3-1=0$, but $a^2+b^2>0$, so $c=1$ does not work.

The prime factorization of 13^3-1 is $2^2 \cdot 3^2 \cdot 61$, which does not contain a prime 3 mod 4 that has an odd exponent. Therefore, $c=13$ is a valid solution.

The prime factorization of 25^3-1 is $2^3 \cdot 3^2 \cdot 7 \cdot 31$, which contains 7^1, so $c=25$ is not a valid solution.

The prime factorization of 37^3-1 is $2^2 \cdot 3^3 \cdot 7 \cdot 67$, which contains 7^1, so $c=37$ is not a valid solution.

The only solution c when $c \equiv 1 \bmod 4$ is 13. We know that 13^3-1 can be expressed as a sum of squares, but we have to make sure that one of the squares isn't 0. However, 13^3-1 is not a perfect square, so a and b are nonzero. Therefore, $c=13$ is a solution.

If c to be 3 mod 4. If $c \equiv 7 \bmod 8$, then $\frac{c-1}{2} \equiv 3 \bmod 4$. Then, $(\frac{c-1}{2})(c^2+c+1) \equiv 3(9+3+1) \equiv 3 \bmod 4$. Therefore, $(\frac{c-1}{2})(c^2+c+1)$ has at least one prime that is 3 mod 4 that has an odd exponent in its prime factorization. As a result, so will $(c-1)(c^2+c+1)$. Therefore, c cannot be 7 mod 8, forcing c to be 3 mod 8.

The only valid c less than or equal to 40 would then be 3, 11, 19, 27, and 35.

The prime factorization of 3^3-1 is $2 \cdot 13$, which does not contain a prime 3 mod 4 that has an odd exponent. Therefore, $c=3$ is a valid solution.

The prime factorization of 11^3-1 is $2 \cdot 5 \cdot 7 \cdot 19$, which contains 7^1, so $c=11$ is not a valid solution.

The prime factorization of 19^3-1 is $2 \cdot 3^3 \cdot 127$, which contains 127^1, so $c=19$ is not a valid solution.

The prime factorization of 27^3-1 is $2 \cdot 13 \cdot 757$, so $c=27$ is a valid solution.

Finally, the prime factorization of 35^3-1 is $2 \cdot 13 \cdot 17 \cdot 97$, which is a valid solution.

The only valid c are 3, 27, and 35, because c^3-1 satisfies the sum of squares theorem. However, a and b are positive integers, so we have to verify that the sum of squares theorem doesn't result in one of a and b being 0. All of 3^3-1, 27^3-1, and 35^3-1 are not perfect squares from looking at their prime factorizations, so all three solutions are valid.

As a result, the sum of all possible values of c is $3+13+27+35=78$. □

11. **[TIEBREAKER]** Estimate the value of

$$\sum_{n=1}^{2023}\left(1+\frac{1}{n}\right)^n$$

to 3 decimal places.

Proposed by Andrew Zhao

Solution. $\boxed{5489.2229}$

□