Theme Round Solutions

LMT Fall 2023

December 16, 2023

Boston Tea Party

Exactly 250 years ago, Boston colonists got angry that the British lowered the price of tea, so they took some tea that the British sent over and dumped it into the Atlantic Ocean. Here are some very true and accurate (fact-checked by Derek) events that took place:

1A. **[6]** Sam dumps tea for 6 hours at a constant rate of 60 tea crates per hour. Eddie takes 4 hours to dump the same amount of tea at a different constant rate. How many tea crates does Eddie dump per hour?

Proposed by Samuel Tsui

Solution. $\boxed{90}$

Sam dumps a total of $6 \cdot 60 = 360$ tea crates. Since it takes Eddie 4 to dump that many, he dumps at a rate of $\frac{360}{4} = 90$ tea crates per hour. □

2A. **[8]** On day 1 of the new year, John Adams and Samuel Adams each drink one gallon of tea. For each positive integer n, on the nth day of the year, John drinks n gallons of tea and Samuel drinks n^2 gallons of tea. After how many days does the combined tea intake of John and Samuel that year first exceed 900 gallons?

Proposed by Aidan Duncan

Solution. $\boxed{13}$

The total amount that John and Samuel consumed by day n is $\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}$. Now, note that our desired number of days should be a bit below $\sqrt[3]{2700}$. Testing a few values gives 13 as our answer. □

3A. **[10]** A rectangular tea bag $PART$ has a logo in its interior at the point Y. The distances from Y to PT and PA are 12 and 9 respectively, and triangles $\triangle PYT$ and $\triangle AYR$ have areas 84 and 42 respectively. Find the perimeter of pentagon $PARTY$.

Proposed by Muztaba Syed

Solution. $\boxed{78}$

Using the area and the height in $\triangle PYT$, we see that $PT = 14$, and thus $AR = 14$, meaning the height from Y to AR is 6. This means $PA = TR = 18$. By the Pythagorean Theorem $PY = \sqrt{12^2 + 9^2} = 15$ and $YT = \sqrt{12^2 + 5^2} = 13$. Combining all of these gives us an answer of $18 + 14 + 18 + 13 + 15 = \boxed{78}$. □

4A. **[12]** Let *Revolution*$(x) = x^3 + Ux^2 + Sx + A$, where U, S, and A are all integers and $U + S + A + 1 = 1773$. Given that *Revolution* has exactly two distinct nonzero integer roots G and B, find the minimum value of $|GB|$.

Proposed by Jacob Xu

Solution. $\boxed{392}$

Notice that $U + S + A + 1$ is just Revolution(1) so Revolution(1) = 1773. Since G and B are integer roots we write Revolution$(X) = (X - G)^2(X - B)$ without loss of generality. So Revolution$(1) = (1 - G)^2(1 - B) = 1773$. 1773 can be factored as $3^2 \times 197$, so to minimize $|GB|$ we set $1 - G = 3$ and $1 - B = 197$. We get that $G = -2$ and $B = -196$ so $|GB| = 392$. □

5A. **[14]** Paul Revere is currently at (x_0, y_0) in the Cartesian plane, which is inside a triangle-shaped ship with vertices at $\left(-\frac{7}{25}, \frac{24}{25}\right)$, $\left(-\frac{4}{5}, -\frac{3}{5}\right)$, and $\left(\frac{4}{5}, -\frac{3}{5}\right)$. Revere has a tea crate in his hands, and there is a second tea crate at $(0,0)$. He must walk to a point on the boundary of the ship to dump the tea, then walk back to pick up the tea crate at the origin. He notices he can take 3 distinct paths to walk the shortest possible distance. Find the ordered pair (x_0, y_0).

Proposed by Derek Zhao

Solution. $\boxed{\left(-\frac{7}{25}, -\frac{6}{25}\right)}$

Let $A = \left(-\frac{7}{25}, \frac{24}{25}\right)$, $B = \left(-\frac{4}{5}, -\frac{3}{5}\right)$, and $C = \left(\frac{4}{5}, -\frac{3}{5}\right)$. Let L, M, and N be the midpoints of BC, AC, and AB, respectively. Let points D, E, and F be the reflections of $O = (0,0)$ over BC, AC, and AB, respectively. Notice since $MN \parallel BC$, $BC \parallel EF$. Therefore, O is the orthocenter of DEF. Notice that (KMN) is the nine-point circle of ABC because it passes through the midpoints and also the nine-point circle of DEF because it passes through the midpoints of the segments connecting a vertex to the orthocenter. Since O is both the circumcenter of ABC and the orthocenter of DEF and the triangles are 180° rotations of each other, Revere is at the orthocenter of ABC. The answer results from adding the vectors $OA + OB + OC$, which gives the orthocenter of a triangle. □

1434

I lost the game.

1B. **[6]** Evaluate $\binom{6}{0} + \binom{6}{1} + \binom{6}{4} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$. Note that $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

Proposed by Jonathan Liu

Solution. $\boxed{64}$

$2^6 = \boxed{64}$ □

2B. **[8]** A four-digit number n is said to be *literally 1434* if, when every digit is replaced by its remainder when divided by 5, the result is 1434. For example, 1984 is literally 1434 because 1 mod 5 is 1, 9 mod 5 is 4, 8 mod 5 is 3, and 4 mod 5 is 4. Find the sum of all 4-digit positive integers that are literally 1434.

Proposed by Evin Liang

Solution. $\boxed{67384}$

The possible numbers are $abcd$ where a is 1 or 6, b is 4 or 9, c is 3 or 8, and d is 4 or 9. There are 16 such numbers and the average is $\frac{8423}{2}$, so the total is 67384. □

3B. **[10]** Evin and Jerry are playing a game with a pile of marbles. On each players‘ turn, they can remove 2, 3, 7, or 8 marbles. If they can't make a move, because there's 0 or 1 marble left, they lose the game. Given that Evin goes first and both players play optimally, for how many values of n from 1 to 1434 does Evin lose the game?

Proposed by Evin Liang

Solution. $\boxed{573}$

Observe that no matter how many marbles a one of them removes, the next player can always remove marbles such that the total number of marbles removed is 10. Thus, when the number of marbles is a multiple of 10, the first player loses the game. We analyse this game based on the number of marbles modulo 10:

If the number of marbles is 0 modulo 10, the first player loses the game If the number of marbles is 2, 3, 7, or 8 modulo 10, the first player wins the game by moving to 0 modulo 10 If the number of marbles is 5 modulo 10, the first player loses the game because every move leads to 2, 3, 7, or 8 modulo 10

In summary, the first player loses if it is 0 mod 5, and wins if it is 2 or 3 mod 5. Now we solve the remaining cases by induction. The first player loses when it is 1 modulo 5 and wins when it is 4 modulo 5. The base case is when there is 1 marble, where the first player loses because there is no move. When it is 4 modulo 5, then the first player can always remove 3 marbles and win by the inductive hypothesis. When it is 1 modulo 5, every move results in 3 or 4 modulo 5, which allows the other player to win by the inductive hypothesis.

Thus, Evin loses the game if n is 0 or 1 modulo 5. There are $\boxed{573}$ such values of n from 1 to 1434. □

4B. **[12]** In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Let M be the midpoint of side AB, G be the centroid of $\triangle ABC$, and E be the foot of the altitude from A to BC. Compute the area of quadrilateral $GAME$.

Proposed by Evin Liang

Solution. $\boxed{23}$

Use coordinates with $A = (0,12)$, $B = (5,0)$, and $C = (-9,0)$. Then $M = \left(\frac{5}{2},6\right)$, $G = \left(-\frac{4}{3},4\right)$, and $E = (0,0)$. By shoelace, the area of $GAME$ is $\boxed{23}$. □

5B. **[14]** Bamal, Halvan, and Zuca are playing *The Game*. To start, they‘re placed at random distinct vertices on regular hexagon $ABCDEF$. Two or more players *collide* when they‘re on the same vertex. When this happens, all the colliding players lose and the game ends. Every second, Bamal and Halvan teleport to a random vertex adjacent to their current position (each with probability $\frac{1}{2}$), and Zuca teleports to a random vertex adjacent to his current position, or to the vertex directly opposite him (each with probability $\frac{1}{3}$). What is the probability that when *The Game* ends Zuca hasn‘t lost?

Proposed by Edwin Zhao

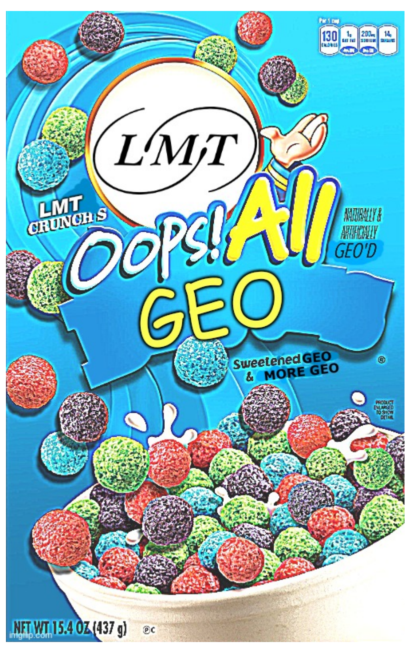
Solution. $\boxed{\frac{29}{90}}$

Color the vertices alternating black and white. By a parity argument if someone is on a different color than the other two they will always win. Zuca will be on opposite parity from the others with probability $\frac{3}{10}$. They will all be on the same parity with probability $\frac{1}{10}$.

At this point there are $2\cdot 2\cdot 3$ possible moves. 3 of these will lead to the same arrangement, so we disregard those. The other 9 moves are all equally likely to end the game. Examining these, we see that Zuca will win in exactly 2 cases (when Bamal and Halvan collide and Zuca goes to a neighboring vertex). Combining all of this, the answer is $\frac{3}{10} + \frac{2}{9}\cdot\frac{1}{10} = \boxed{\frac{29}{90}}$ □

Oops! All Geo

Oops! We seem to have accidentally written a lot of geometry questions. Or did we?



1C. **[6]** How many distinct triangles are there with prime side lengths and perimeter 100?

Proposed by Muztaba Syed

Solution. $\boxed{0}$

As the perimeter is even, 1 of the sides must be 2. Thus, the other 2 sides are congruent by Triangle Inequality. Thus, for the perimeter to be 100, both of the other sides must be 49, but as 49 is obviously composite, the answer is thus $\boxed{0}$ □

2C. **[8]** Let R be the rectangle on the cartesian plane with vertices $(0,0)$, $(5,0)$, $(5,7)$, and $(0,7)$. Find the number of squares with sides parallel to the axes and vertices that are lattice points that lie within the region bounded by R.

Proposed by Boyan Litchev

Solution. $\boxed{85}$

We have $(6-n)(8-n)$ distinct squares with side length n, so the total number of squares is $5\cdot7+4\cdot6+3\cdot5+2\cdot4+1\cdot3 = 35+24+15+8+3 = \boxed{85}$. □

3C. **[10]** Determine the least integer n such that for any set of n lines in the 2-D plane, there exists either a subset of 1001 lines that are all parallel, or a subset of 1001 lines that are pairwise nonparallel.

Proposed by Samuel Wang

Solution. $\boxed{1000001}$

Since being parallel is a transitive property, we note that in order for this to not exist, there must exist at most 1001 groups of lines, all pairwise intersecting, with each group containing at most 1001 lines. Thus, $n = 1000^2+1 = \boxed{1000001}$ □

4C. **[12]** The equation of line ℓ_1 is $24x-7y = 319$ and the equation of line ℓ_2 is $12x-5y = 125$. Let a be the number of positive integer values of n less than 2023 such that for both ℓ_1 and ℓ_2 there exists a lattice point on that line that is a distance of n from the point $(20,23)$. Determine a.

Proposed by Christopher Cheng

Solution. $\boxed{6}$

Note that $(20,23)$ is the intersection of the lines ℓ_1 and ℓ_2. Thus, we only care about lattice points on the the two lines that are an integer distance away from $(20,23)$. Notice that 7 and 24 are part of the Pythagorean triple $(7,24,25)$ and 5 and 12 are part of the Pythagorean triple $(5,12,13)$. Thus, points on ℓ_1 only satisfy the conditions when n is divisible by 25 and points on ℓ_2 only satisfy the conditions when n is divisible by 13. Therefore, a is just the number of numbers that are divisible by both 25 and 13. The LCM of 25 and 13 is 325, and n is less than 2023 so the answer is $\boxed{6}$. □

5C. **[14]** In equilateral triangle ABC, $AB = 2$ and M is the midpoint of AB. A laser is shot from M in a certain direction. Whenever the laser collides with a side of ABC, it reflects off that side such that the acute angle formed by the incident ray and the side is equal to the acute angle formed by the reflected ray and the side. When the laser coincides with a vertex, it stops. Find the sum of the least three possible integer distances that the laser could have traveled.

Proposed by Jerry Xu

Solution. $\boxed{21}$

Solution 1: Whenever the laser hits a side of the triangle, reflect the laser's path over that side so that the path of the laser forms a straight line. We want the path of the laser to coincide with a vertex of one of the reflected triangles. Thus, we can restate the problem as follows:

Tessellate the plane with equilateral triangles of side length 3. Consider one of these equilateral triangles ABC with M being the midpoint of $AB = 2$. Find the sum of the three minimum integer distances from M to any vertex in the plane.

It is trivial to see that the vertical distance between M and a given vertex is $n\sqrt{3}$ for $n \in \mathbb{N}^0$. If n is even, the horizontal distance between O and a given vertex is $1+2m$ for $m \in \mathbb{N}^0$. If n is odd, the horizontal distance is $2m$ for $m \in \mathbb{N}^0$. We consider two separate cases:

Case 1: n is even. We thus want to find $l \in \mathbb{N}$ such that

$$\left(n\sqrt{3}\right)^2 + (1+2m)^2 = l^2.$$

Make the substitution $1+2m = k$ to get that

$$3n^2 + k^2 = l^2.$$

Notice that these equations form a family of generalized Pell equations $y^2 - 3x^2 = N$ with $N = k^2$. We can find some set of roots to these equations using the multiplicative principle: we will use this idea to find three small l values, and that gives us an upper bound on what the three l values can be. From there, a simple bash of lower l values to see if solutions to each generalized Pell equation not given by the multiplicative principle exist finishes this case.

By the multiplicative principle some set of solutions (x_n, y_n) to the above equation with sufficiently small x_n follow the formula

$$x_n\sqrt{3} + y_n = \left(x_0\sqrt{3} + y_0\right)\left(u_n\sqrt{3} + v_n\right),$$

where $\left(x_0, y_0\right)$ is a solution to the generalized Pell equation and (u_n, v_n) are solutions to the Pell equation $y^2 - 3x^2 = 1$. Remember that the solutions to this last Pell equation satisfy

$$u_n\sqrt{3} + v_n = \left(u_0\sqrt{3} + v_0\right)^k$$

where the trivial positive integer solution

$$(u_0, v_0) = (1, 2)$$

(this can easily be found by inspection or by taking the convergents of the continued fraction expansion of $\sqrt{3}$).

We thus get that

$$(u_1, v_1) = (4, 7), (u_2, v_2) = (15, 26), (u_2, v_2) = (56, 97)\ldots$$

(also don't forget that $(u, v) = (0, 1)$ is another solution).

From here, note that k must be odd since $k = 1+2m$ for $m \in \mathbb{N}^0$. For $k = 1$, the smallest three solutions to the Pell equation with n even are

$$(x, y) = (0, 1), (4, 7), (56, 97)$$
$$\longrightarrow (n, m, l) = (0, 0, 1), (4, 0, 7), (56, 0, 97)$$

Our current smallest three values of l are thus $1, 7, 97$. A quick check confirms that all of these solutions are not extraneous (extraneous solutions appear when the path taken by the laser prematurely hits a vertex).

For $k = 3$, using the multiplicative principle we get two new smaller solutions

$$(x, y) = (0, 3), (12, 21)$$
$$\longrightarrow (n, m, l) = (0, 1, 3), (12, 1, 21)$$

However, note that $(n, m, l) = (0, 1, 3)$ is extraneous since is equivalent to the path that is traced out by the solution $(n, m, l) = (0, 0, 1)$ found previously and will thus hit a vertex prematurely. Thus, our new three smallest values of l are $1, 7, 21$.

For $k \geq 5$, it is evident that there are no more smaller integral values of l that can be found using the multiplicative principle: the solution set $(n, m, l) = \left(0, \dfrac{k-1}{2}, k\right)$ is always extraneous for $k > 1$ since it is equivalent to the path traced out by $(0, 0, 1)$ as described above, and any other solutions will give larger values of l.

Thus, we now only need to consider solutions to each generalized Pell equation not found by the multiplicative principle. A quick bash shows that $l = 3, 5, 9, 11$ gives no solutions for any odd k and even n, however $n = 13$ gives $k = 11$ and $n = 4$, a non-extraneous solution smaller than one of the three we currently have. Thus, our new three smallest l values are $1, 7, 13$. Case 2: n is odd. We thus want to find $l \in \mathbb{N}$ such that

$$\left(n\sqrt{3}\right)^2 + (2m)^2 = l^2.$$

Make the substitution $2m = k$ to get that

$$3n^2 + k^2 = l^2.$$

This is once again a family of generalized Pell equations with $N = k^2$, however this time we must have k even instead of k odd. However, note that there are no solutions to this family of Pell equation with n odd: $k^2 \equiv 0 \pmod 4$ since k is even, and $3n^2 \equiv 3 \pmod 4$ since n is odd, however $0+3 \equiv 3 \pmod 4$ is not a possible quadratic residue mod 4. Thus, this case gives no solutions. Our final answer is thus $1+7+13 = \boxed{21}$.

Solution 2: Reflecting the triangle over its sides instead of the laser in the coordinate plane with M=(0,0), we get each point as $(x, y\sqrt{3})$ for integers x, y. We thus want the smallest integers z such that x and y exist such that $x^2 + 3y^2 = z^2$, or $3y^2 = z^2 - x^2 = (z+x)(z-x)$. We get that as exactly one of y, x is odd, z is odd, giving even y and odd x. We now casework on y: $y = 0$: $z = 1, x = 1$ $y = 2$: nothing works. $y = 4$: only $z = 7$, $x = 1$ and $z = 13$, $x = 11$ Thus, the smallest values of z are $1, 7, 13$, for $1+7+13 = \boxed{21}$ □

Tiebreaker Estimation

This problem will only be used to break ties for individual aggregate awards. If two tied competitors submit distinct estimates a_1 and a_2, the competitor who submitted a_1 wins if $\left|\log \frac{a_1}{c}\right| < \left|\log \frac{a_2}{c}\right|$ where c is the correct answer. Otherwise, the competitor who answered a_2 wins.

1. **[Tiebreaker]** In Lexington, each year lasts 202320232023 days and each day is equally likely to be a given person's birthday. Sam gathers n random people from Lexington in a room such that there is at least a 50% chance that there exists a pair of people who share a birthday. What is the least possible integer value of n?

   Proposed by Sam Wang

   Solution. $\boxed{529598}$

   Birthday Problem. Can be approximated as $\sqrt{202320232023} \approx 449800$, which would have been among the closest estimates. □