Theme Round

LMT Fall 2023

December 16, 2023

Boston Tea Party

Exactly 250 years ago, Boston colonists got angry that the British lowered the price of tea, so they took some tea that the British sent over and dumped it into the Atlantic Ocean. Here are some very true and accurate (fact-checked by Derek) events that took place:

1A. **[6]** Sam dumps tea for 6 hours at a constant rate of 60 tea crates per hour. Eddie takes 4 hours to dump the same amount of tea at a different constant rate. How many tea crates does Eddie dump per hour?

2A. **[8]** On day 1 of the new year, John Adams and Samuel Adams each drink one gallon of tea. For each positive integer n, on the nth day of the year, John drinks n gallons of tea and Samuel drinks n^2 gallons of tea. After how many days does the combined tea intake of John and Samuel that year first exceed 900 gallons?

3A. **[10]** A rectangular tea bag $PART$ has a logo in its interior at the point Y. The distances from Y to PT and PA are 12 and 9 respectively, and triangles $\triangle PYT$ and $\triangle AYR$ have areas 84 and 42 respectively. Find the perimeter of pentagon $PARTY$.

4A. **[12]** Let $\mathit{Revolution}(x) = x^3 + Ux^2 + Sx + A$, where U, S, and A are all integers and $U + S + A + 1 = 1773$. Given that *Revolution* has exactly two distinct nonzero integer roots G and B, find the minimum value of $|GB|$.

5A. **[14]** Paul Revere is currently at (x_0, y_0) in the Cartesian plane, which is inside a triangle-shaped ship with vertices at $\left(-\frac{7}{25}, \frac{24}{25}\right)$, $\left(-\frac{4}{5}, -\frac{3}{5}\right)$, and $\left(\frac{4}{5}, -\frac{3}{5}\right)$. Revere has a tea crate in his hands, and there is a second tea crate at $(0,0)$. He must walk to a point on the boundary of the ship to dump the tea, then walk back to pick up the tea crate at the origin. He notices he can take 3 distinct paths to walk the shortest possible distance. Find the ordered pair (x_0, y_0).

1434

I lost the game.

1B. **[6]** Evaluate $\binom{6}{0} + \binom{6}{1} + \binom{6}{4} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$. Note that $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

2B. **[8]** A four-digit number n is said to be *literally 1434* if, when every digit is replaced by its remainder when divided by 5, the result is 1434. For example, 1984 is literally 1434 because 1 mod 5 is 1, 9 mod 5 is 4, 8 mod 5 is 3, and 4 mod 5 is 4. Find the sum of all 4-digit positive integers that are literally 1434.

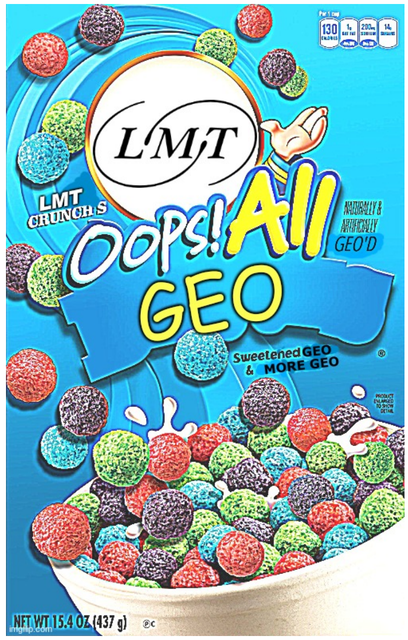
3B. **[10]** Evin and Jerry are playing a game with a pile of marbles. On each players‘ turn, they can remove 2, 3, 7, or 8 marbles. If they can't make a move, because there's 0 or 1 marble left, they lose the game. Given that Evin goes first and both players play optimally, for how many values of n from 1 to 1434 does Evin lose the game?

4B. **[12]** In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Let M be the midpoint of side AB, G be the centroid of $\triangle ABC$, and E be the foot of the altitude from A to BC. Compute the area of quadrilateral $GAME$.

5B. **[14]** Bamal, Halvan, and Zuca are playing *The Game*. To start, they‘re placed at random distinct vertices on regular hexagon $ABCDEF$. Two or more players *collide* when they‘re on the same vertex. When this happens, all the colliding players lose and the game ends. Every second, Bamal and Halvan teleport to a random vertex adjacent to their current position (each with probability $\frac{1}{2}$), and Zuca teleports to a random vertex adjacent to his current position, or to the vertex directly opposite him (each with probability $\frac{1}{3}$). What is the probability that when *The Game* ends Zuca hasn‘t lost?

Oops! All Geo

Oops! We seem to have accidentally written a lot of geometry questions. Or did we?



1C. **[6]** How many distinct triangles are there with prime side lengths and perimeter 100?

2C. **[8]** Let R be the rectangle on the cartesian plane with vertices $(0,0)$, $(5,0)$, $(5,7)$, and $(0,7)$. Find the number of squares with sides parallel to the axes and vertices that are lattice points that lie within the region bounded by R.

3C. **[10]** Determine the least integer n such that for any set of n lines in the 2-D plane, there exists either a subset of 1001 lines that are all parallel, or a subset of 1001 lines that are pairwise nonparallel.

4C. **[12]** The equation of line ℓ_1 is $24x-7y=319$ and the equation of line ℓ_2 is $12x-5y=125$. Let a be the number of positive integer values of n less than 2023 such that for both ℓ_1 and ℓ_2 there exists a lattice point on that line that is a distance of n from the point $(20,23)$. Determine a.

5C. **[14]** In equilateral triangle ABC, $AB=2$ and M is the midpoint of AB. A laser is shot from M in a certain direction. Whenever the laser collides with a side of ABC, it reflects off that side such that the acute angle formed by the incident ray and the side is equal to the acute angle formed by the reflected ray and the side. When the laser coincides with a vertex, it stops. Find the sum of the least three possible integer distances that the laser could have traveled.

Tiebreaker Estimation

This problem will only be used to break ties for individual aggregate awards. If two tied competitors submit distinct estimates a_1 and a_2, the competitor who submitted a_1 wins if $\left|\log \frac{a_1}{c}\right| < \left|\log \frac{a_2}{c}\right|$ where c is the correct answer. Otherwise, the competitor who answered a_2 wins.

1. **[Tiebreaker]** In Lexington, each year lasts 202320232023 days and each day is equally likely to be a given person's birthday. Sam gathers n random people from Lexington in a room such that there is at least a 50% chance that there exists a pair of people who share a birthday. What is the least possible integer value of n?