Team Round Solutions

LMT Fall 2023

December 16, 2023

1. **[16]** George has 150 cups of flour and 200 eggs. He can make a cupcake with 3 cups of flour and 2 eggs, or he can make an omelet with 4 eggs. What is the maximum number of treats (both omelets and cupcakes) he can make?

   Proposed by Jacob Xu

   Solution. $\boxed{75}$

   To maximize ingredient usage, make 50 cupcakes to use all of the flour. There will be 100 eggs left to make 25 omelets. So he can make $\boxed{75}$ total treats. □

2. **[18]** For how many nonnegative integer values of k does the equation $7x^2 + kx + 11 = 0$ have no real solutions?

   Proposed by Ethan Xu

   Solution. $\boxed{18}$

   $k^2 - (4)(7)(11) < 0$ from the discriminant. Simplifying gives $k^2 < 308$. Thus, $0 \le k < \sqrt{308}$, as we are looking for nonnegative integer values. From this inequality, we see that the possible values of k are the integers from 0 to 17, inclusive, which includes $\boxed{18}$ numbers. □

3. **[20]** Adam and Topher are playing a game in which each of them starts with 2 pickles. Each turn, they flip a fair coin: if it lands heads, Topher takes 1 pickle from Adam; if it lands tails, Adam takes 2 pickles from Topher. (If Topher has only 1 pickle left, Adam will just take it.) What's the probability that Topher will have all 4 pickles before Adam does?

   Proposed by Christopher Cheng

   Solution. $\boxed{\frac{2}{7}}$

   H represents a heads, and T represents a tails Cases:

   T - Adam wins, probability $\frac{1}{2}$.

   HH - Topher wins, probability $\frac{1}{4}$.

   HTT - Adam wins, probability $\frac{1}{4}$.

   HTH - we return to the original setup of 2 pickles vs 2 pickles, probability $\frac{1}{8}$.

   These probabilities add up to 1, so these are our only cases.

   Let the answer to this problem be p. We want the probability that Topher wins, so we don't count when Adam wins. From the above cases, we get $p = 0 \cdot (\frac{1}{2} + \frac{1}{8}) + 1 \cdot \frac{1}{4} + \frac{1}{8} \cdot p$ after solving, we get $\boxed{p = \frac{2}{7}}$ □

4. **[22]** Fred chooses a positive two-digit number with distinct nonzero digits. Laura takes Fred's number and swaps its digits. She notices that the sum of her number and Fred's number is a perfect square and the positive difference between them is a perfect cube. Find the greater of the two numbers.

   Proposed by Jacob Xu

Solution. $\boxed{74}$

The sum must be a multiple of 11, and since the sum is a square, that sum must be 121. We see that 47 and 74 have a sum of 121 and a difference of 27. So the answer is $\boxed{74}$. □

5. **[24]** In regular hexagon $ABCDEF$ with side length 2, let P, Q, R, and S be the feet of the altitudes from A to BC, EF, CF, and BE, respectively. Find the area of quadrilateral $PQRS$.

Proposed by Muztaba Syed

Solution. $\boxed{\frac{9\sqrt{3}}{4}}$

This consists of 3 equilateral triangles (APS, ASR, ARQ) each with side length $\sqrt{3}$ meaning our answer is $\boxed{\frac{9\sqrt{3}}{4}}$. □

6. **[26]** Jeff rolls a standard 6 sided die repeatedly until he rolls either all of the prime numbers possible at least once, or all the of even numbers possible at least once. Find the probability that his last roll is a 2.

Proposed by Muztaba Syed

Solution. $\boxed{\frac{7}{15}}$

Disregard the 1s and any repeated rolls. Then we have a random permutation of $\{2,3,4,5,6\}$ with each being equally likely, and we want the 2 to appear after the 3 and 5 or after the 4 and 6. We can do this by looking at the position of the 2. If it is the 3rd number, there are $2 \cdot 2 \cdot 2$ ways, and if it is 4th or 5th then there are 4! ways. This gives us a probability of $\frac{8+24+24}{5!} = \boxed{\frac{7}{15}}$. □

7. **[28]** How many 2-digit factors does 555555 have?

Proposed by Zachary Perry

Solution. $\boxed{12}$

The prime factorization of $555555 = 5 \cdot 111 \cdot 1001$ is $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. There are 3 prime 2-digit factors and 9 2-digit factors that are products of 2 prime factors $\{15, 21, 33, 39, 35, 55, 65, 77, 91\}$. There are $\boxed{12}$ in total. □

8. **[30]** Let J, E, R, and Y be four positive integers chosen independently and uniformly at random from the set of factors of 1428. What is the probability that $JERRY = 1428$? Express your answer in the form $\frac{a}{b \cdot 2^n}$ where n is a nonnegative integer, a and b are odd, and $\gcd(a,b) = 1$.

Proposed by Evin Liang

Solution. $\boxed{\frac{7}{3 \cdot 2^{12}}}$

The prime factorization of 1428 is $2^2 \cdot 3 \cdot 7 \cdot 17$. This means that there are 24^4 possibilities. Also, either $R = 1$ or $R = 2$.

Case 1: $R = 1$. Then there are $6 \cdot 3 \cdot 3 \cdot 3$ possibilities.

Case 2: $R = 2$. Then there are $1 \cdot 3 \cdot 3 \cdot 3$ possibilities.

Consequently, $p = \boxed{\frac{7}{3 \cdot 2^{12}}}$. □

9. **[32]** In triangle ABC, let O be the circumcenter and let G be the centroid. The line perpendicular to OG at O intersects BC at M such that M, G, and A are collinear and $OM = 3$. Compute the area of ABC, given that $OG = 1$.

Proposed by Evin Liang

Solution. $\boxed{54}$

Since M, G, and A are collinear M must be the midpoint of BC. Also since $OM \perp BC$, OG must be parallel to BC. Then we can find $GA = 2\sqrt{10}$ from which we get $OA = 3\sqrt{5}$ by Law of Cosines on $\triangle AGO$. This means that $MB = MC = \sqrt{45-3^2} = 6$. Additionally the height from A is 3 times the height from P which is 9. This means the area is $12 \cdot 9 \cdot \frac{1}{2} = \boxed{54}$ □

10. **[34]** Aidan and Andrew independently select distinct cells in a 4 by 4 grid, as well as a direction (either up, down, left, or right), both at random. Every second, each of them will travel 1 cell in their chosen direction. Find the probability that Aidan and Andrew will meet (be in the same cell at the same time) before either one of them hits an edge of the grid. (If Aidan and Andrew cross paths by switching cells, it doesn't count as meeting.)

Proposed by Muztaba Syed

Solution. $\boxed{\frac{3}{80}}$

In total there are $16 \cdot 15 \cdot 4^2$ ways to arrange them. If they are in the same row/column there are $8 \cdot 2 \cdot 2$ ways to arrange them. Otherwise they will be at 2 vertices of a 2 by 2, 3 by 3, or 4 by 4 square. The number of ways for each of these will be $9 \cdot 4 \cdot 2$, $4 \cdot 4 \cdot 2$, and $4 \cdot 2$. Putting these together we get an answer of $\boxed{\frac{3}{80}}$ □

11. **[38]** Find the number of degree 8 polynomials $f(x)$ with nonnegative integer coefficients satisfying both $f(1) = 16$ and $f(-1) = 8$.

Proposed by Jerry Xu

Solution. $\boxed{47775}$

Say $f(x) = a_8x^8 + a_7x^7 + \cdots + a_1x + a_0$. We thus have that

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 = 16,$$
$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 = 8.$$

Subtracting the second equation from the first (or, alternatively, adding the two equations would give a synonymous equation), we get that

$$2a_1 + 2a_3 + 2a_5 + 2a_7 = 8 \longrightarrow a_1 + a_3 + a_5 + a_7 = 4.$$

Since a_1, a_3, a_5, a_7 are nonnegative integers, by stars and bars we have that there are $\binom{4+4-1}{4-1} = 35$ distinct ordered pairs (a_1, a_3, a_5, a_7) that satisfy this condition. Since $a_1 + a_3 + a_5 + a_7 = 4$, from the first equation we have that

$$a_0 + a_2 + a_4 + a_6 + a_8 = 12$$

But a_0 can't be 0, so let $a_0' = a_0 - 1$. Then we have

$$a_0' + a_2 + a_4 + a_6 + a_8 = 11$$

Which has $\binom{11+5-1}{5-1} = \binom{15}{4}$ ordered sets that work. We thus have a final answer of $\binom{7}{3} \cdot \binom{15}{4} = \boxed{47775}$. □

12. **[42]** In triangle ABC with $AB = 7$, $AC = 8$, and $BC = 9$, the A-excircle is tangent to BC at point D and also tangent to lines AB and AC at points E and F, respectively. Find $[DEF]$. (The A-excircle is the circle tangent to segment BC and the extensions of rays AB and AC. Also, $[XYZ]$ denotes the area of triangle XYZ.)

Proposed by Jerry Xu

Solution. $\boxed{\frac{80\sqrt{5}}{7}}$

Note that $[DEF] = [AEF] - [ABC] - [BDE] - [CDF]$. We trivially have by Heron's that

$$[ABC] = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = 12\sqrt{5}.$$

By excircle properties, we have that $BE = s - b = 12 - 7 = 5$ and $CF = s - c = 12 - 8 = 4$, where s is the semiperimeter of ABC (this can be seen by applying equal tangents, where we find $AE = AF$, $BE = BD$ and $CF = CD$). Thus, $AE = AF = 12$. By Law of Cosines, we have that

$$9^2 = 8^2 + 7^2 + 2 \cdot 8 \cdot 7 \cdot \cos\angle BAC,$$

so $\cos\angle BAC = \frac{2}{7}$. Thus, $\sin\angle BAC = \frac{3\sqrt{5}}{7}$. By sine area, we have that

$$[AEF] = \frac{1}{2} \cdot 12^2 \cdot \frac{3\sqrt{5}}{7} = \frac{216\sqrt{5}}{7}.$$

Note that $\sin\angle EBC = \sin 180^\circ - \angle ABC = \sin\angle ABC$. By Law of Cosines again, we have that $\cos\angle ABC = \frac{11}{21}$ so $\sin\angle ABC = \frac{8\sqrt{5}}{21}$. By sine area again, we have that

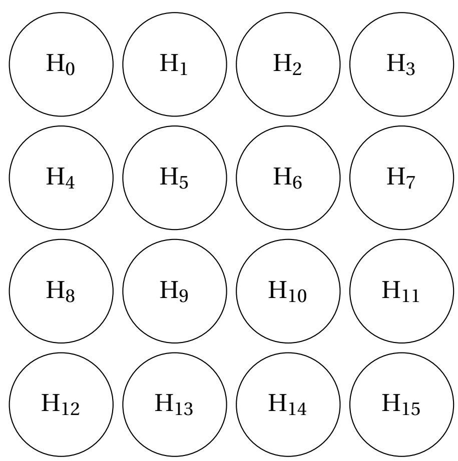
$$[BDE] = \frac{1}{2} \cdot 5^2 \cdot \frac{8\sqrt{5}}{21} = \frac{100\sqrt{5}}{21}$$

(remember that $BD = BE = 5$). Applying the same process to CDF gives us that $[CDF] = \frac{8\sqrt{5}}{3}$. Our final answer is

$$\frac{216\sqrt{5}}{7} - 12\sqrt{5} - \frac{100\sqrt{5}}{21} - \frac{8\sqrt{5}}{3} = \boxed{\frac{80\sqrt{5}}{7}}.$$

□

13. **[46]** Ella lays out 16 coins heads up in a 4×4 grid as shown.



On a move, Ella can flip all the coins in any row, column, or diagonal (including small diagonals such as H_1 & H_4). If rotations are considered distinct, how many distinct grids of coins can she create in a finite number of moves?

Proposed by Atticus Oliver

Solution. $\boxed{32768}$

Notice that there are 8 edge coins (those on the edges excluding corners). Every time, we flip exactly 0 or 2 edge coins. Therefore, there are always an even number of edge coins on heads and an even number on tails, for an upper bound of $2^8\left(\binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} + \binom{8}{3}\right) = 2^{15} = \boxed{32768}$.

Now we show this is achievable. It's possible to toggle the state of any arbitrary corner or center, as well as toggling any two arbitrary edges at the same time, and therefore any state that has an even number of heads up can be reached by flipping the pieces using the below algorithms. For the rest of the "proof", consider the grid of coins to be labelled with the rows from top to bottom being A1, A2, A3, A4; followed by B1, B2, B3, B4; followed by C1, C2, C3, C4; followed by D1, D2, D3, D4.

Consider flipping just the B2 center piece. This can be done by flipping the A3-C1 diagonal, the A2-B1 diagonal, the A4-A4 diagonal, the D1-D1 diagonal, the A row, and the 1 column. A similar argument can be provided for flipping other center pieces. Consider flipping just edge pieces A2 and A3. This can be done by flipping the A row, the A1-A1 diagonal, and the A4-A4 diagonal. A similar argument can be provided for flipping other edges that are orthogonally adjacent.

Consider flipping just edge pieces A2 and B4. This can be done by first flipping both A2 and A3 as above and then flipping the A3-B4 diagonal. A similar argument can be provided for other edges a knight's move apart.

Consider flipping just edge pieces A2 and C4. This can be done by first flipping A2 and A3 as above, then flipping B4 and C4 as above, then flipping the A3-B4 diagonal. A similar argument can be provided for other edges an alfil's move apart.

Consider flipping just edge pieces A2 and D3. This can be done by first flipping A2 and C4 as above, then flipping the C4-D3 diagonal. A similar argument can be provided for other pieces a camel's move apart. Consider flipping just edge pieces A2 and D2. This can be done by first flipping the 2 column, then flipping the B2 and C2 center pieces as above. A similar argument can be provided for other pieces a threeleaper's move apart.

Lastly, the proof for flipping the A1 corner and the A2 and B1 edges simultaneously (and their rotations) has been left as an exercise to the reader. □

14. **[50]** Find $\sum_{i=1}^{100} i \gcd(i, 100)$.

Proposed by William Hua

Solution. $\boxed{31000}$

$S = \sum_{i=1}^{100} i \gcd(i,100) = \sum_{i=0}^{100} i \gcd(i,100) = \sum_{i=0}^{100} i \gcd(100-i,100) = \sum_{i=0}^{100} (100-i)\gcd(i,100)$ $S+S = \sum_{i=0}^{100} 100\gcd(i,100) = 100\sum_{i=0}^{100} \gcd(i,100)$

Prime factorization of 100 is $2^2 \cdot 5^2$.

$\sum_{i=0}^{100} \gcd(i,100) = 100 + \sum_{i=1}^{100} \gcd(i,100) = 100 + (4+1+2+1)(25+1+1+1+1+5+1+1+1+1+5+1+1+1+1+5+1+1+1+1+5+1+1+1+1) = 100 + 8 \cdot 65 = 620$.

Answer is $\frac{100 \cdot 620}{2} = \boxed{31000}$. □

15. **[54]** In triangle ABC with $AB = 26$, $BC = 28$, and $CA = 30$, let M be the midpoint of AB and let N be the midpoint of CA. The circumcircle of triangle BCM intersects AC at $X \neq C$, and the circumcircle of triangle BCN intersects AB at $Y \neq B$. Lines MX and NY intersect BC at P and Q, respectively. The area of quadrilateral $PQYX$ can be expressed as $\frac{p}{q}$ for positive integers p and q such that $\gcd(p,q) = 1$. Find q.

Proposed by William Hua

Solution. $\boxed{12675}$

Angle chase w/ cyclic quads yields $\angle XMN = \angle YNM$, so $PM \| QN$, so $PQNM$ is a parallelogram. This means $PQ = MN = 14$. The altitude of the triangle from A to BC is 24, so the altitude of the parallelogram is 12. Area of parallelogram is $14 \cdot 12$.

Then, $[PQYX] = [PQNM] + [MXY] - [MNY]$.

By power of a point, $AX = \frac{169}{15}$ and $AY = \frac{225}{13}$. Then, $MY = \frac{56}{13}$. Then, $[MYN] = \frac{1}{2} \cdot \frac{MY}{AB} \cdot [ABC] = \frac{14 \cdot 336}{169}$.

Also, $[MXY] = \frac{AX}{AN} \cdot [MNY] = \frac{169}{225}[MNY] = \frac{14 \cdot 336}{225}$.

Therefore, $[PQYX] = 14 \cdot 12 + \frac{14 \cdot 336}{225} - \frac{14 \cdot 336}{169}$. To find the denominator of this number, note that the $14 \cdot 12$ is irrelevant. Then, $\frac{1}{225} - \frac{1}{169}$ has a denominator of $225 \cdot 169$, so $14 \cdot 336 \cdot \left(\frac{1}{225} - \frac{1}{169}\right)$ has a denominator of $\frac{225 \cdot 169}{3} = \boxed{12675}$. □