Team Round

LMT Fall 2023

December 16, 2023

1. **[16]** George has 150 cups of flour and 200 eggs. He can make a cupcake with 3 cups of flour and 2 eggs, or he can make an omelet with 4 eggs. What is the maximum number of treats (both omelets and cupcakes) he can make?

2. **[18]** For how many nonnegative integer values of k does the equation $7x^2 + kx + 11 = 0$ have no real solutions?

3. **[20]** Adam and Topher are playing a game in which each of them starts with 2 pickles. Each turn, they flip a fair coin: if it lands heads, Topher takes 1 pickle from Adam; if it lands tails, Adam takes 2 pickles from Topher. (If Topher has only 1 pickle left, Adam will just take it.) What's the probability that Topher will have all 4 pickles before Adam does?

4. **[22]** Fred chooses a positive two-digit number with distinct nonzero digits. Laura takes Fred's number and swaps its digits. She notices that the sum of her number and Fred's number is a perfect square and the positive difference between them is a perfect cube. Find the greater of the two numbers.

5. **[24]** In regular hexagon $ABCDEF$ with side length 2, let P, Q, R, and S be the feet of the altitudes from A to BC, EF, CF, and BE, respectively. Find the area of quadrilateral $PQRS$.

6. **[26]** Jeff rolls a standard 6 sided die repeatedly until he rolls either all of the prime numbers possible at least once, or all the of even numbers possible at least once. Find the probability that his last roll is a 2.

7. **[28]** How many 2-digit factors does 555555 have?

8. **[30]** Let J, E, R, and Y be four positive integers chosen independently and uniformly at random from the set of factors of 1428. What is the probability that $JERRY = 1428$? Express your answer in the form $\frac{a}{b\cdot 2^n}$ where n is a nonnegative integer, a and b are odd, and $\gcd(a,b) = 1$.

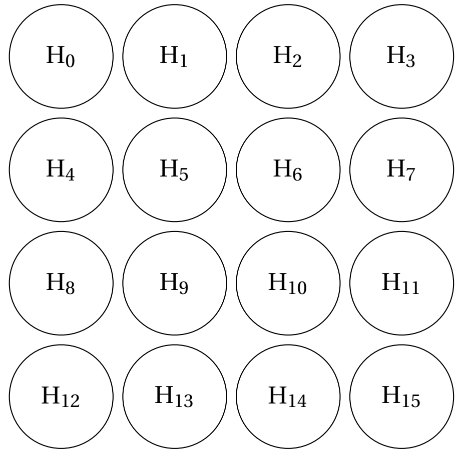
9. **[32]** In triangle ABC, let O be the circumcenter and let G be the centroid. The line perpendicular to OG at O intersects BC at M such that M, G, and A are collinear and $OM = 3$. Compute the area of ABC, given that $OG = 1$.

10. **[34]** Aidan and Andrew independently select distinct cells in a 4 by 4 grid, as well as a direction (either up, down, left, or right), both at random. Every second, each of them will travel 1 cell in their chosen direction. Find the probability that Aidan and Andrew will meet (be in the same cell at the same time) before either one of them hits an edge of the grid. (If Aidan and Andrew cross paths by switching cells, it doesn't count as meeting.)

11. **[38]** Find the number of degree 8 polynomials $f(x)$ with nonnegative integer coefficients satisfying both $f(1) = 16$ and $f(-1) = 8$.

12. **[42]** In triangle ABC with $AB = 7$, $AC = 8$, and $BC = 9$, the A-excircle is tangent to BC at point D and also tangent to lines AB and AC at points E and F, respectively. Find $[DEF]$. (The A-excircle is the circle tangent to segment BC and the extensions of rays AB and AC. Also, $[XYZ]$ denotes the area of triangle XYZ.)

13. **[46]** Ella lays out 16 coins heads up in a 4×4 grid as shown.



On a move, Ella can flip all the coins in any row, column, or diagonal (including small diagonals such as H_1 & H_4). If rotations are considered distinct, how many distinct grids of coins can she create in a finite number of moves?

14. **[50]** Find $\sum_{i=1}^{100} i \gcd(i, 100)$.

15. **[54]** In triangle ABC with $AB = 26$, $BC = 28$, and $CA = 30$, let M be the midpoint of AB and let N be the midpoint of CA. The circumcircle of triangle BCM intersects AC at $X \neq C$, and the circumcircle of triangle BCN intersects AB at $Y \neq B$. Lines MX and NY intersect BC at P and Q, respectively. The area of quadrilateral $PQYX$ can be expressed as $\frac{p}{q}$ for positive integers p and q such that $\gcd(p, q) = 1$. Find q.