Speed Round Solutions

LMT Fall 2023

December 16, 2023

1. **[6]** Define the operator $\diamondsuit$ in the following way: For positive integers a and b, $a\diamondsuit b = |a-b| \cdot |b-a|$. Evaluate $1\diamondsuit(2\diamondsuit(3\diamondsuit(4\diamondsuit 5)))$.

   Proposed by Muztaba Syed

   Solution. $\boxed{9}$

   $a\diamondsuit b = (a-b)^2$. This gives us an answer of $\boxed{9}$. □

2. **[6]** Eddie has a study block that lasts 1 hour. It takes Eddie 25 minutes to do his homework and 5 minutes to play a game of Clash Royale. He can't do both at the same time. How many games can he play in this study block while still completing his homework?

   Proposed by Edwin Zhao

   Solution. $\boxed{7}$

   Study block lasts 60 minutes, thus he has 35 minutes to play games, during which he can play $\frac{35}{5} = \boxed{7}$ games. □

3. **[6]** Sam Wang decides to evaluate an expression of the form $x+2\times 2+y$. However, he unfortunately reads each 'plus' as a 'times' and reads each 'times' as a 'plus'. Surprisingly, he still gets the problem correct. Find $x+y$.

   Proposed by Edwin Zhao

   Solution. $\boxed{4}$

   We have $x+2\times 2+y = x\times 2+2\times y$. When simplifying, we have $x+y+4 = 2x+2y$, and $x+y=4$. □

4. **[6]** The numbers 1, 2, 3, and 4 are randomly arranged in a 2×2 grid with one number in each cell. Find the probability the sum of two numbers in the top row of the grid is even.

   Proposed by Muztaba Syed and Derek Zhao

   Solution. $\boxed{\frac{1}{3}}$

   Pick a number for the top-left. There is one number that makes the sum even no matter what we pick. Therefore, the answer is $\boxed{\frac{1}{3}}$. □

5. **[6]** Let a and b be two-digit positive integers. Find the greatest possible value of $a+b$, given that the greatest common factor of a and b is 6.

   Proposed by Jacob Xu

   Solution. $\boxed{186}$

   We can write our two numbers as $6x$ and $6y$. Notice that x and y must be relatively prime. Since $6x$ and $6y$ are two digit numbers, we just need to check values of x and y from 2 through 16 such that x and y are relatively prime. We maximize the sum when $x=15$ and $y=16$, since consecutive numbers are always relatively prime. So the sum is $6*(15+16) = 186$. □

6. **[6]** Blue rolls a fair n-sided die that has sides its numbered with the integers from 1 to n, and then he flips a coin. Blue knows that the coin is weighted to land heads either $\frac{1}{3}$ or $\frac{2}{3}$ of the time. Given that the probability of both rolling a 7 and flipping heads is $\frac{1}{15}$, find n.

   Proposed by Jacob Xu

   Solution. $\boxed{10}$

   The chance of getting any given number is $\frac{1}{n}$, so the probability of getting 7 and heads is either $\frac{1}{n} * \frac{1}{3} = \frac{1}{3n}$ or $\frac{1}{n} * \frac{2}{3} = \frac{2}{3n}$. So we get that either $n = 5$ or $n = 10$, but since rolling a 7 is possible, then only $n = 10$ is a solution. □

7. **[6]** Isabella is making sushi. She slices a piece of salmon into the shape of a solid triangular prism. The prism is 2cm thick, and its triangular faces have side lengths 7cm, 24cm, and 25cm. Find the volume of this piece of salmon in cm^3.

   Proposed by Isabella Li

   Solution. $\boxed{168}$

   $V = \text{area of base} \times \text{height} = 7 \times 24 \div 2 \times 2 = \boxed{168}$. □

8. **[6]** To celebrate the 20th LMT, the LHS Math Team bakes a cake. Each of the n bakers places 20 candles on the cake. When they count, they realize that there are $(n-1)!$ total candles on the cake. Find n.

   Proposed by Christopher Cheng

   Solution. $\boxed{6}$

   First notice that n is really small. This is obvious as if $n = 10$ then 200 candles are placed on the cake when there are supposed to be 9! or 362880 candles. From here, we can just guess and check or we can realize that if $20n$ is a factor of a factorial, that factorial must have a factor of 5 and 4, and thus must also have a factor of 3, 2, and 1 so n must be divisible by 6. We already established that $n < 10$ and the only positive multiple of 6 less than 10 is 6 itself. Plugging in $\boxed{n = 6}$, we see that it satisfies the conditions. □

9. **[6]** Find the least positive integer k such that when $\frac{k}{2023}$ is written in simplest form, the sum of the numerator and denominator is divisible by 7.

   Proposed by Muztaba Syed

   Solution. $\boxed{35}$

   If k is not divisible by 7, then the denominator is divisible by 7 so it isn't possible. So then let $k = 7a$, so the fraction is $\frac{a}{289}$, so we see the smallest possible a is 5. Then $k = \boxed{35}$ □

10. **[6]** A square has vertices $(0,10)$, $(0,0)$, $(10,0)$, and $(10,10)$ on the x-y coordinate plane. A second quadrilateral is constructed with vertices $(0,10)$, $(0,0)$, $(10,0)$, and $(15,15)$. Find the positive difference between the areas of the original square and the second quadrilateral.

    Proposed by William Hua

    Solution. $\boxed{50}$

    New shape is $225 - 2 \cdot (5 \cdot 15)/2 = 150$. Old is 100. Answer is $\boxed{50}$. □

11. **[6]** Let $LEXINGT_1ONMAT_2H$ be a regular 13-gon. Find $\angle LMT_1$, in degrees.

    Proposed by Edwin Zhao

    Solution. $\boxed{\frac{1080}{13}}$

    Note that this polygon is cyclic, thus $\angle LMT_1$ is half the angle of the arc LT_1 not containing M, which is $\frac{1}{2} \times 6 \times \frac{360}{13} = \boxed{\frac{1080}{13}}$. □

12. **[6]** Sam and Jonathan play a game where they take turns flipping a weighted coin, and the game ends when one of them wins. The coin has a $\frac{8}{9}$ chance of landing heads and a $\frac{1}{9}$ chance of landing tails. Sam wins when he flips heads, and Jonathan wins when he flips tails. Find the probability that Sam wins, given that he takes the first turn.

    Proposed by Samuel Tsui

    Solution. $\boxed{\frac{72}{73}}$

    The probability that Sam wins on the first flip is $\frac{8}{9}$. However if he does not win on the first flip the probability that he wins on the third flip is $\frac{1}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}$. Noticing this is an infinite geometric sequence with first term $\frac{8}{9}$ and ratio $\frac{1}{9}\cdot\frac{8}{9}$ the probability Sam wins is $\frac{\frac{8}{9}}{1-\frac{8}{81}} = \boxed{\frac{72}{73}}$. □

13. **[6]** Given that the base-17 integer $\overline{8323a02421_{17}}$ (where a is a base-17 digit) is divisible by $\overline{16_{10}}$, find a. Express your answer in base 10.

    Proposed by Jonathan Liu

    Solution. $\boxed{7}$

    Similar to divisibility of 9 in base 10, divisibility by 16 in base 17 requires the sum of the digits to be divisible by 16 (easy proof by mod 16) so $8+3+2+3+a+0+2+4+2+1 = 25+a$, and this result must be 32, so $\boxed{a=7}$ □

14. **[6]** In obtuse triangle ABC with $AB = 7$, $BC = 20$, and $CA = 15$, let point D be the foot of the altitude from C to line AB. Evaluate $[ACD]+[BCD]$. (Note that $[XYZ]$ means the area of triangle XYZ.)

    Proposed by Jonathan Liu

    Solution. $\boxed{150}$

    Since triangle ABC is obtuse, point D lies outside the triangle, and then you can use the Pythagorean theorem on ACD and BCD to find that $AD = 9$ and $CD = 12$. So, $[ACD] = \frac{9\cdot 12}{2}$ and $[BCD] = \frac{16\cdot 12}{2}$ so the answer is $\boxed{150}$. □

15. **[6]** Find the least positive integer n greater than 1 such that $n^3 - n^2$ is divisible by $7^2 \times 11$.

    Proposed by Jacob Xu

    Solution. $\boxed{56}$

    $n^3 - n^2 = (n^2)(n-1)$. This must be divisible by 7^2, so we should test values where either n is divisible by 7 or $n-1$ is divisible by 49. We test 7, 14, 21, 28, 35, 42, 49, 50, and we get $n = 56$ works □

16. **[6]** Jeff writes down the two-digit base-10 prime $\overline{ab_{10}}$. He realizes that if he misinterprets the number as the base 11 number $\overline{ab_{11}}$ or the base 12 number $\overline{ab_{12}}$, it is still a prime. What is the least possible value of Jeff's number (in base 10)?

    Proposed by Muztaba Syed

    Solution. $\boxed{61}$

    The numbers are $10a+b$, $11a+b$, and $12a+b$. Looking at this mod 2 and 3, we see that $6 \mid a$ because otherwise one of the numbers will be divisible by 2 or 3. Then checking $a = 6$ gives $\boxed{61}$ works. □

17. **[6]** Samuel Tsui and Jason Yang each chose a different integer between 1 and 60, inclusive. They don't know each others' numbers, but they both know that the other person's number is between 1 and 60 and distinct from their own. They have the following conversation:

    Samuel Tsui: Do our numbers have any common factors greater than 1?

    Jason Yang: Definitely not. However their least common multiple must be less than 2023.

    Samuel Tsui: Ok, this means that the sum of the factors of our two numbers are equal.

    What is the sum of Samuel Tsui's and Jason Yang's numbers?

    Proposed by Samuel Tsui

Solution. $\boxed{52}$

From what Jason said we know that his number must be a prime greater than 30. However since the lcm must be less than 2023, his number must be 31 because if it was 37 and Sam's number was 59 the lcm would be greater than 2023. From Sam's last statement his number must be 21 as the sum of the factors of 31 is 32 and 21 is the only other number with the sum of the factors as 32. This means the sum of Sam's and Jason's numbers is $31+21 = \boxed{52}$. □

18. **[6]** In square $ABCD$ with side length 2, let M be the midpoint of AB. Let N be a point on AD such that $AN = 2ND$. Let point P be the intersection of segment MN and diagonal AC. Find the area of triangle BPM.

Proposed by Jacob Xu

Solution. $\boxed{\frac{2}{7}}$

The segment $AM = 1$ and the segment $AN = \frac{4}{3}$. So the $[AMN]$ is $\frac{1}{2}\cdot 1\cdot\frac{4}{3} = \frac{2}{3}$. Notice that the heights from P to AM and AN are the same, so the ratio of $[APN]$ to $[APM]$ is $4:3$. So $[APM] = \frac{3}{7}\cdot\frac{2}{3} = \frac{2}{7}$. Finally, since $AM = MB$ and triangles APM and BPM share the same height, the triangles have the same area. So $[BPM] = \boxed{\frac{2}{7}}$. □

19. **[6]** Evin picks distinct points A, B, C, D, E, and F on a circle. What is the probability that there are exactly two intersections among the line segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$?

Proposed by Evin Liang

Solution. $\boxed{\frac{1}{5}}$

It only matters what the order of the points is. Additionally, fix A in place. Now, there are 3 cases for where B can be:

Case 1: right next to A. This case is impossible, as then AB does not intersect either CD or EF.

Case 2: 2 points away from A. Thus, any point can be between A and B and that line intersects AB and the other will not. WLOG, let C be between A and B. Thus, AB intersects CD but not EF. Thus, CD and EF must intersect. In this case, D must be opposite C. Thus, the probability in this case is $\frac{2}{5}\cdot\frac{1}{3} = \frac{2}{15}$

Case 3: opposite A. This means that AB either intersects both of CD and EF or neither. Clearly, it must intersect both, thus the lines CD and EF are set, up to rearrangement. The probability of this case is $\frac{1}{5}\cdot\frac{1}{3} = \frac{1}{15}$

The total probability is thus $\frac{2+1}{15} = \boxed{\frac{1}{5}}$ □

20. **[6]** The remainder when $x^{100} - x^{99} + \cdots - x + 1$ is divided by $x^2 - 1$ can be written in the form $ax + b$. Find $2a + b$.

Proposed by Calvin Garces

Solution. $\boxed{-49}$

By polynomial remainder theorem we can plug in $x = 1$ and $x = -1$ to get the values of $-a + b$ and $a + b$. $x = 1$ gives $a + b = 1$ and $x = -1$ gives $-a + b = 101$. Solving gives $a = -50$ and $b = 51$, so the answer is $-100 + 51 = \boxed{-49}$. □

21. **[6]** Let $(a_1, a_2, a_3, a_4, a_5)$ be a random permutation of the integers from 1 to 5 inclusive. Find the expected value of

$$\sum_{i=1}^{5} |a_i - i| = |a_1 - 1| + |a_2 - 2| + |a_3 - 3| + |a_4 - 4| + |a_5 - 5|.$$

Proposed by Muztaba Syed

Solution. $\boxed{8}$

The expected distance from a_1 to 1 is $\frac{0+1+2+3+4}{5} = 2$, and the same for a_5. For a_2 and a_4 the expected distance is $\frac{1+0+1+2+3}{5} = \frac{7}{5}$. For a_3 the expected distance from 3 is $\frac{2+1+0+1+2}{5} = \frac{6}{5}$. We can add these to get $2 + 2 + \frac{7}{5} + \frac{7}{5} + \frac{6}{5} = \boxed{8}$. □

22. **[6]** Consider all pairs of points (a,b,c) and (d,e,f) in the 3-D coordinate system with $ad+be+cf=-2023$. What is the least positive integer that can be the distance between such a pair of points?

Proposed by William Hua

Solution. $\boxed{90}$

The distance squared is equal to $(a-d)^2+(b-e)^2+(c-f)^2$. Expanding yields $(a^2+b^2+c^2+d^2+e^2+f^2)-2(ad+be+cf) = (a^2+b^2+c^2+d^2+e^2+f^2)+4046$.

However, by AM-GM, $\frac{a^2+d^2}{2} \ge |ad|$. Similarly, $\frac{b^2+e^2}{2} \ge |be|$ and $\frac{c^2+f^2}{2} \ge |cf|$. Then, $a^2+b^2+c^2+d^2+e^2+f^2 \ge 2(|ad|+|be|+|cf|) \ge 2(|ad+be+cf|) = 4046$. Equality holds when $a=b=c=-d=-e=-f=\sqrt{\frac{2023}{3}}$.

Therefore, $(a-d)^2+(b-e)^2+(c-f)^2 \ge 8092$.

Now, notice that if we keep $ad=-\frac{2023}{3}$ constant, the value of a^2+d^2 lies in $[\frac{4046}{3},+\infty)$. This means $(a-d)^2+(b-e)^2+(c-f)^2 \in [8092,+\infty)$.

The smallest positive integer distance would then be $\boxed{90}$, as $89^2 < 8092$ but $90^2 \ge 8092$. □

23. **[6]** Let S be the set of all positive integers n such that the sum of all factors of n, including 1 and n, is 120. Compute the sum of all numbers in S.

Proposed by Evin Liang

Solution. $\boxed{292}$

The sum of divisors function is multiplicative. The possible prime powers dividing n are 2, 8, 3, 27, 5, 7, 11, 19, 23, 29, and 59. The sum of divisors of each are 3, 15, 4, 40, 6, 8, 12, 20, 24, 30, and 60. Then checking these possibilities we see n can be $3\cdot 29$, $5\cdot 19$, $2^3\cdot 7$, and $2\cdot 3^3$ which gives a total of $87+95+56+54=\boxed{292}$. □

24. **[6]** Evaluate

$$2023\cdot\frac{2023^6+27}{(2023^2+3)(2024^3-1)}-2023^2.$$

Proposed by Evin Liang

Solution. $\boxed{-6066}$

By sum of cubes we see,

$$a^6+27b^6=(a^2+3b^2)(a^2-3ab+3b^2)(a^2+3ab+3b^2)$$

So $2023^6+27=(2023^2+3)(2023^2-3\cdot 2023+3)(2023+3\cdot 2023+3)$. Since $\frac{2024^3-1}{2023}=2023^2+3\cdot 2023+3$, $2023\cdot\frac{2023^6+27}{(2023^2+3)(2024^3-1)}-2023^2=2023^2-3\cdot 2023+3-2023^2=\boxed{-6066}$. □

25. **[6]** In triangle ABC with centroid G and circumcircle ω, line $\overline{AG}$ intersects BC at D and ω at P. Given that $GD=DP=3$, and $GC=4$, find AB^2.

Proposed by Muztaba Syed

Solution. $\boxed{168}$

Using centroid properties, $AG=2\cdot GD=6$. Then by Power of a Point at D, $AD\cdot DP=BD\cdot CD=9$, which means that $BD=CD=3\sqrt{3}$. Now we can compute PC, either with the Law of Cosines or by Parallelogram Law on $BGCP$. Then

$$2PC^2+2BP^2=GP^2+BC^2$$

Using $GC=4$, $GP=6$, and $BC=6\sqrt{3}$, we get that $PC=2\sqrt{14}$. Then since $\triangle ADB\sim\triangle CDP$ with ratio $\sqrt{3}$, our answer is $2\sqrt{14}\cdot\sqrt{3}\Rightarrow\boxed{168}$. □