Speed Round

LMT Fall 2023

December 16, 2023

1. **[6]** Define the operator $\diamondsuit$ in the following way: For positive integers a and b, $a\diamondsuit b = |a-b|\cdot|b-a|$. Evaluate $1\diamondsuit(2\diamondsuit(3\diamondsuit(4\diamondsuit 5)))$.

2. **[6]** Eddie has a study block that lasts 1 hour. It takes Eddie 25 minutes to do his homework and 5 minutes to play a game of Clash Royale. He can't do both at the same time. How many games can he play in this study block while still completing his homework?

3. **[6]** Sam Wang decides to evaluate an expression of the form $x+2\times 2+y$. However, he unfortunately reads each 'plus' as a 'times' and reads each 'times' as a 'plus'. Surprisingly, he still gets the problem correct. Find $x+y$.

4. **[6]** The numbers 1, 2, 3, and 4 are randomly arranged in a 2×2 grid with one number in each cell. Find the probability the sum of two numbers in the top row of the grid is even.

5. **[6]** Let a and b be two-digit positive integers. Find the greatest possible value of $a+b$, given that the greatest common factor of a and b is 6.

6. **[6]** Blue rolls a fair n-sided die that has sides its numbered with the integers from 1 to n, and then he flips a coin. Blue knows that the coin is weighted to land heads either $\frac{1}{3}$ or $\frac{2}{3}$ of the time. Given that the probability of both rolling a 7 and flipping heads is $\frac{1}{15}$, find n.

7. **[6]** Isabella is making sushi. She slices a piece of salmon into the shape of a solid triangular prism. The prism is 2cm thick, and its triangular faces have side lengths 7cm, 24cm, and 25cm. Find the volume of this piece of salmon in cm^3.

8. **[6]** To celebrate the 20th LMT, the LHS Math Team bakes a cake. Each of the n bakers places 20 candles on the cake. When they count, they realize that there are $(n-1)!$ total candles on the cake. Find n.

9. **[6]** Find the least positive integer k such that when $\frac{k}{2023}$ is written in simplest form, the sum of the numerator and denominator is divisible by 7.

10. **[6]** A square has vertices $(0,10)$, $(0,0)$, $(10,0)$, and $(10,10)$ on the x-y coordinate plane. A second quadrilateral is constructed with vertices $(0,10)$, $(0,0)$, $(10,0)$, and $(15,15)$. Find the positive difference between the areas of the original square and the second quadrilateral.

11. **[6]** Let $LEXINGT_1ONMAT_2H$ be a regular 13-gon. Find $\angle LMT_1$, in degrees.

12. **[6]** Sam and Jonathan play a game where they take turns flipping a weighted coin, and the game ends when one of them wins. The coin has a $\frac{8}{9}$ chance of landing heads and a $\frac{1}{9}$ chance of landing tails. Sam wins when he flips heads, and Jonathan wins when he flips tails. Find the probability that Sam wins, given that he takes the first turn.

13. **[6]** Given that the base-17 integer $\overline{8323a02421}_{17}$ (where a is a base-17 digit) is divisible by $\overline{16}_{10}$, find a. Express your answer in base 10.

14. **[6]** In obtuse triangle ABC with $AB=7$, $BC=20$, and $CA=15$, let point D be the foot of the altitude from C to line AB. Evaluate $[ACD]+[BCD]$. (Note that $[XYZ]$ means the area of triangle XYZ.)

15. **[6]** Find the least positive integer n greater than 1 such that n^3-n^2 is divisible by $7^2\times 11$.

16. **[6]** Jeff writes down the two-digit base-10 prime $\overline{ab}_{10}$. He realizes that if he misinterprets the number as the base 11 number $\overline{ab}_{11}$ or the base 12 number $\overline{ab}_{12}$, it is still a prime. What is the least possible value of Jeff's number (in base 10)?

17. **[6]** Samuel Tsui and Jason Yang each chose a different integer between 1 and 60, inclusive. They don't know each others' numbers, but they both know that the other person's number is between 1 and 60 and distinct from their own. They have the following conversation:

    Samuel Tsui: Do our numbers have any common factors greater than 1?

    Jason Yang: Definitely not. However their least common multiple must be less than 2023.

    Samuel Tsui: Ok, this means that the sum of the factors of our two numbers are equal.

    What is the sum of Samuel Tsui's and Jason Yang's numbers?

18. **[6]** In square $ABCD$ with side length 2, let M be the midpoint of AB. Let N be a point on AD such that $AN = 2ND$. Let point P be the intersection of segment MN and diagonal AC. Find the area of triangle BPM.

19. **[6]** Evin picks distinct points A, B, C, D, E, and F on a circle. What is the probability that there are exactly two intersections among the line segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$?

20. **[6]** The remainder when $x^{100} - x^{99} + \cdots - x + 1$ is divided by $x^2 - 1$ can be written in the form $ax + b$. Find $2a + b$.

21. **[6]** Let $(a_1, a_2, a_3, a_4, a_5)$ be a random permutation of the integers from 1 to 5 inclusive. Find the expected value of

$$\sum_{i=1}^{5} |a_i - i| = |a_1 - 1| + |a_2 - 2| + |a_3 - 3| + |a_4 - 4| + |a_5 - 5|.$$

22. **[6]** Consider all pairs of points (a, b, c) and (d, e, f) in the 3-D coordinate system with $ad + be + cf = -2023$. What is the least positive integer that can be the distance between such a pair of points?

23. **[6]** Let S be the set of all positive integers n such that the sum of all factors of n, including 1 and n, is 120. Compute the sum of all numbers in S.

24. **[6]** Evaluate

$$2023 \cdot \frac{2023^6 + 27}{(2023^2 + 3)(2024^3 - 1)} - 2023^2.$$

25. **[6]** In triangle ABC with centroid G and circumcircle ω, line $\overline{AG}$ intersects BC at D and ω at P. Given that $GD = DP = 3$, and $GC = 4$, find AB^2.