Speed Round

Lexington High School

December 11th, 2021

1. **[6]** Compute $21 \cdot 21 - 20 \cdot 20$.

2. **[6]** A square has side length 2. If the square is scaled by a factor of n, the perimeter of the new square is equal to the area of the original square. Find $10n$.

3. **[6]** Kevin has 2 red marbles and 2 blue marbles in a box. He randomly grabs two marbles. The probability that they are the same color can be expressed as $\frac{a}{b}$ for relatively prime integers a and b. Find $a+b$.

4. **[6]** In a classroom, if the teacher splits the students into groups of 3 or 4, there is one student left out. If the students form groups of 5, every student is in a group. What is the fewest possible number of students in this classroom?

5. **[6]** Find the sum of all positive integer values of x such that $\lfloor \sqrt{x!} \rfloor = x$.

6. **[6]** Find the number of positive integer factors of

$$2021^{(2^0+2^1)} \cdot 1202^{(1^2+0^2)}.$$

7. **[6]** Let n be the number of days over a 13 year span. Find the difference between the greatest and least possible values of n. Note: All years divisible by 4 are leap years unless they are divisible by 100 but not 400. For example, 2000 and 2004 are leap years, but 1900 is not.

8. **[6]** In isosceles $\triangle ABC$, $AB = AC$, and $\angle ABC = 72^\circ$. The bisector of $\angle ABC$ intersects AC at D. Given that $BC = 30$, find AD.

9. **[6]** For an arbitrary positive value of x, let h be the area of a regular hexagon with side length x and let s be the area of a square with side length x. Find the value of $\left\lfloor \frac{10h}{s} \right\rfloor$.

10. **[6]** There is a half-full tub of water with a base of 4 inches by 5 inches and a height of 8 inches. When an infinitely long stick with base 1 inch by 1 inch is inserted vertically into the bottom of the tub, the number of inches the water level rises by can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

11. **[6]** Find the sum of all 4-digit numbers with digits that are a permutation of the digits in 2021. Note that positive integers cannot have first digit 0.

12. **[6]** A 10-digit base 8 integer is chosen at random. The probability that it has 30 digits when written in base 2 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

13. **[6]** Call a natural number *sus* if it can be expressed as k^2+k+1 for some positive integer k. Find the sum of all sus integers less than 2021.

14. **[6]** In isosceles triangle ABC, D is the intersection of AB and the perpendicular to BC through C. Given that $CD = 5$ and $AB = BC = 1$, find $\sec^2 \angle ABC$.

15. **[6]** Every so often, the minute and hour hands of a clock point in the same direction. The second time this happens after 1:00 is $\frac{a}{b}$ minutes later, where a and b are relatively prime positive integers. Find $a+b$.

16. **[6]** The 999-digit number $N = 123123\ldots123$ is composed of 333 iterations of the number 123. Find the least nonnegative integer m such that $N+m$ is a multiple of 101.

17. **[6]** The sum of the reciprocals of the divisors of 2520 can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

18. **[6]** Duncan, Paul, and 6 Atreides guards are boarding three helicopters. Duncan, Paul, and the guards enter the helicopters at random, with the condition that Duncan and Paul do not enter the same helicopter. Note that not all helicopters must be occupied. The probability that Paul has more guards with him in his helicopter than Duncan does can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

19. **[6]** Let the minimum possible distance from the origin to the parabola $y = x^2 - 2021$ be d. The value of d^2 can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

20. **[6]** In quadrilateral $ABCD$ with interior point E and area $49\sqrt{3}$, $\frac{BE}{CE} = \frac{2}{\sqrt{3}}$, $\angle ABC = \angle BCD = 90°$, and $\triangle ABC \sim \triangle BCD \sim \triangle BEC$. The length of AD can be expressed as $\sqrt{n}$ where n is a positive integer. Find n.

21. **[6]** Find the value of

$$\sum_{i=1}^{\infty}\left(\frac{i^2}{2^{i-1}}+\frac{i^2}{2^i}+\frac{i^2}{2^{i+1}}\right)=\left(\frac{1^2}{2^0}+\frac{1^2}{2^1}+\frac{1^2}{2^2}\right)+\left(\frac{2^2}{2^1}+\frac{2^2}{2^2}+\frac{2^2}{2^3}\right)+\left(\frac{3^2}{2^2}+\frac{3^2}{2^3}+\frac{3^2}{2^4}\right)+\cdots.$$

22. **[6]** Five not necessarily distinct digits are randomly chosen in some order. Let the probability that they form a nondecreasing sequence be $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the remainder when $a+b$ is divided by 1000.

23. **[6]** Real numbers a, b, c, and d satisfy

$$ac - bd = 33$$
$$ad + bc = 56.$$

Given that $a^2 + b^2 = 5$, find the sum of all possible values of $c^2 + d^2$.

24. **[6]** Jeff has a fair tetrahedral die with sides labeled 0, 1, 2, and 3. He continuously rolls the die and record the numbers rolled in that order. For example, if he rolls a 1, then rolls a 2, and then rolls a 3, he writes down 123. He keeps rolling the die until he writes the substring 2021. What is the expected number of times he rolls the die?

25. **[6]** In triangle ABC, $BC = 2\sqrt{3}$, and $AB = AC = 4\sqrt{3}$. Circle ω with center O is tangent to segment AB at T, and ω is also tangent to ray CB past B at another point. Points O, T, and C are collinear. Let r be the radius of ω. Given that $r^2 = \frac{a}{b}$ where a and b are relatively prime positive integers, find $a+b$.