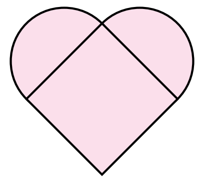
4th Annual Fall Lexington Math Tournament - Guts Round - Part 1

Team Name: ________________________________

________________ 1. **[9]** The temperature inside is $28°F$. After the temperature is increased by $5°C$, what will the new temperature in Fahrenheit be?

________________ 2. **[9]** Find the least positive integer value of n such that $\sqrt{2021+n}$ is a perfect square.

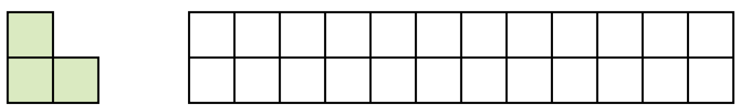
________________ 3. **[9]** A heart consists of a square with two semicircles attached by their diameters as shown in the diagram. Given that one of the semicircles has a diameter of length 10, then the area of the heart can be written as $a+b\pi$ where a and b are positive integers. Find $a+b$.



4th Annual Fall Lexington Math Tournament - Guts Round - Part 2

Team Name: ________________________________

________________ 4. **[10]** An L-shaped tromino is a group of 3 blocks (where blocks are squares) arranged in a L shape, as pictured below to the left. How many ways are there to fill a 12 by 2 rectangle of blocks (pictured below to the right) with L-shaped trominos if the trominos can be rotated or reflected?



________________ 5. **[10]** How many permutations of the word PIKACHU are there such that no two vowels are next to each other?

________________ 6. **[10]** Find the number of primes n such that there exists another prime p such that both $n+p$ and $n-p$ are also prime numbers.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 3

Team Name: ________________________________

________________ 7. **[11]** Maisy the Bear is at the origin of the Cartesian Plane. When Maisy is on the point (m,n) then it can jump to either $(m,n+1)$ or $(m+1,n)$. Let $L(x,y)$ be the number of jumps it takes for Maisy to reach point (x,y). The sum of $L(x,y)$ over all lattice points (x,y) with both coordinates between 0 and 2020, inclusive, is denoted as S. Find $\frac{S}{2020}$.

________________ 8. **[11]** A circle with center O and radius 2 and a circle with center P and radius 3 are externally tangent at A. Points B and C are on the circle with center O such that $\triangle ABC$ is equilateral. Segment AB extends past B to point D and AC extends past C to point E such that $BD = CE = \sqrt{3}$. The area of $\triangle DEP$ can be written as $\frac{a\sqrt{b}}{c}$ where a, b, and c are integers such that b is squarefree and $\gcd(a,c)=1$. Find $a+b+c$.

________________ 9. **[11]** Find the number of trailing zeroes at the end of

$$\prod_{i=1}^{2021}(2021+i-1) = (2021)(2022)\cdots(4041).$$

4th Annual Fall Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[12]** Let a, b, and c be side lengths of a rectangular prism with space diagonal 10. Find the value of

$$(a+b)^2+(b+c)^2+(c+a)^2-(a+b+c)^2.$$

______________ 11. **[12]** In a regular heptagon $ABCDEFG$, ℓ is a line through E perpendicular to DE. There is a point P on ℓ outside the heptagon such that $PA = BC$. Find the measure of $\angle EPA$.

______________ 12. **[12]** Dunan is being "SUS". The word "SUS" is a palindrome. Find the number of palindromes that can be written using some subset of the letters {S, U, S, S, Y, B, A, K, A}.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 5

Team Name: ______________________________

______________ 13. **[13]** Jason flips a coin repeatedly. The probability that he flips 15 heads before flipping 4 tails can be expressed as $\frac{a}{2^b}$ where a and b are positive integers and a is odd. Find $a+b$.

______________ 14. **[13]** Triangle ABC has side lengths $AB = 3$, $BC = 3$, and $AC = 4$. Let D be the intersection of the angle bisector of $\angle BAC$ and segment BC. Let the circumcircle of $\triangle BAD$ intersect segment AC at a point E distinct from A. The length of AE can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

______________ 15. **[13]** The sum of the squares of all values of x such that $\{(x-2)(x-3)\} = \{(x-1)(x-6)\}$ and $\lfloor x^2-6x+6\rfloor = 9$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Note: $\{a\}$ is the fractional part function, and returns $a-\lfloor a\rfloor$.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 6

Team Name: ______________________________

Déjà Vu?

______________ 16. **[14]** Maisy the Polar Bear is at the origin of the Polar Plane ($r = 0, \theta = 0$). Maisy's location can be expressed as (r,θ), meaning it is a distance of r away from the origin and at a angle of θ degrees counterclockwise from the x-axis. When Maisy is on the point (m,n) then it can jump to either $(m,n+1)$ or $(m+1,n)$. Maisy cannot jump to any point it has been to before. Let $L(r,\theta)$ be the number of paths Maisy can take to reach point (r,θ). The sum of $L(r,\theta)$ over all points where r is an integer between 1 and 2020 and θ is an integer between 0 and 359 can be written as $\frac{n^k-1}{m}$ for some minimum value of n, such that n, k, and m are all positive integers. Find $n+k+m$.

______________ 17. **[14]** A circle with center O and radius 2 and a circle with center P and radius 3 are externally tangent at A. Points B and C are on the circle with center O such that $\triangle ABC$ is equilateral. Segment AB extends past B to point D and AC extends past C to point E such that $BD = CE = \sqrt{3}$. A line through D is tangent to circle P at F. Find DF^2.

______________ 18. **[14]** Find the number of trailing zeroes at the end of

$$\prod_{i=1}^{2021}(2021i-1) = (2020)(4041)\ldots(2021^2-1).$$

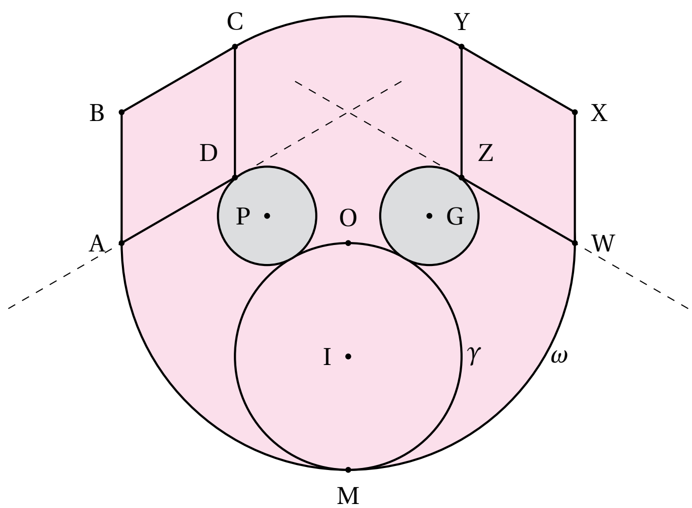
Team Name: ______________________________

______________ 19. **[15]** A function $f(n)$ is defined as follows:

$$f(n)=\begin{cases}\frac{n}{3} & \text{if } n\equiv 0 \pmod 3\\ n^2+4n-5 & \text{if } n\equiv 1 \pmod 3\\ n^2+n-2 & \text{if } n\equiv 2 \pmod 3\end{cases}$$

Find the number of integer values of n between 2 and 1000 inclusive such that $f(f(\cdots f(n))) = 1$ for some number of applications of $f(n)$.

______________ 20. **[15]** In the diagram below, the larger circle with diameter AW has radius 16. $ABCD$ and $WXYZ$ are rhombi where $\angle BAD = \angle XWZ = 60^\circ$ and $AC = CY = YW$. M is the midpoint of minor arc AW, as shown. Let I be the center of the circle with diameter OM. Circles with center P and G are tangent to lines AD and WZ, respectively, and also tangent to the circle with center I. Given that $IP \perp AD$ and $IG \perp WZ$, the area of $\triangle PIG$ can be written as $a + b\sqrt{c}$ where a, b, and c are positive integers and c is not divisible by the square of a prime. Find $a + b + c$.



______________ 21. **[15]** In a list of increasing consecutive positive integers, the first item is divisible by 1, the second item is divisible by 4, the third item is divisible by 7, and this pattern increases up to the seventh item being divisible by 19. Find the remainder when the least possible value of the first item in the list is divided by 100.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[17]** Let the answer to Problem 24 be C. Jacob never drinks more than C cups of coffee in a day. He always drinks a positive integer number of cups. The probability that he drinks $C+1-X$ cups is X times the probability he drinks C cups of coffee for any positive number X from 1 to C inclusive. Find the expected number of cups of coffee he drinks.

______________ 23. **[17]** Let the answer to Problem 22 be A. Three lines are drawn intersecting the interior of a triangle with side lengths 26, 28, and 30 such that each line is parallel and a distance A away from a respective side. The perimeter of the triangle formed by the three new lines can be expressed as $\frac{a}{b}$ for relatively prime integers a and b. Find $a+b$.

______________ 24. **[17]** Let the answer to Problem 23 be B. Given that $ab-c = bc-a = ca-b$ and $a^2+b^2+c^2 = B+2$, find the sum of all possible values of $|a+b+c|$.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

Déjà Déjà Vu?

______________ 25. **[20]** Maisy the Bear is at the origin of the Cartesian Plane. When Maisy is on the point (m,n) then it can jump to either $(m,n+1)$ or $(m+1,n)$. Let $L(x,y)$ be the number of paths Maisy can take to reach the point (x,y). The sum of $L(x,y)$ over all lattice points (x,y) with both coordinates between 0 and 2020, inclusive, can be written as $\binom{2k}{k}-j$ for a minimum positive integer k and corresponding positive integer j. Find $k+j$.

______________ 26. **[20]** A circle with center O and radius 2 and a circle with center P and radius 3 are externally tangent at A. Points B and C are on the circle with center O such that $\triangle ABC$ is equilateral. Segment AB extends past B to point D and AC extends past C to point E such that $BD = CE = \sqrt{3}$. A line through D is tangent to circle P at F. The value of EF^2 can be expressed as $\frac{a+b\sqrt{c}}{d}$ where a,b,c, and d are integers, c is squarefree, and $\gcd(a,b,d) = 1$. Find $a+b+c+d$.

______________ 27. **[20]** Find the number of trailing zeroes at the end of

$$\prod_{i=1}^{2021}(2021^i - 1) = (2021^1 - 1)\dots(2021^{2021} - 1).$$

4th Annual Fall Lexington Math Tournament - Guts Round - Part 10

Team Name: ______________________________

______________ 28. **[23]** Points A, B, C, P, and D lie on circle ω in that order. Let AC and BD intersect at I. Given that $PI = PC = PD$, $\angle DAB = 137°$, and $\angle ABC = 109°$, find the measure of $\angle BIC$ in degrees.

______________ 29. **[23]** Find the sum of all positive integers $n < 2021$ such that when $\{d_1, d_2, \dots, d_k\}$ are the positive integer factors of n, then

$$\left(\sum_{i=1}^{k} d_i\right)\cdot\left(\sum_{i=1}^{k}\frac{1}{d_i}\right) = r^2$$

for some rational number r.

______________ 30. **[23]** Let a, b, c, d and e be positive real numbers. Define the function $f(x,y) = \frac{x}{y}+\frac{y}{x}$ for all positive real numbers. Given that $f(a,b) = 7$, $f(b,c) = 5$, $f(c,d) = 3$, and $f(d,e) = 2$, find the sum of all possible values of $f(e,a)$.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 11

Team Name: ______________________________

______________ 31. **[26]** There exist 100 (not necessarily distinct) complex numbers $r_1, r_2, \ldots, r_{100}$ such that for any positive integer $1 \le k \le 100$, we have that $P(r_k) = 0$ where the polynomial P is defined as

$$P(x) = \sum_{i=1}^{101} i \cdot x^{101-i} = x^{100} + 2x^{99} + 3x^{98} + \ldots + 99x^2 + 100x + 101.$$

Find the value of

$$\prod_{j=1}^{100} (r_j^2 + 1) = (r_1^2 + 1)(r_2^2 + 1)\ldots(r_{100}^2 + 1).$$

______________ 32. **[26]** Let BT be the diameter of a circle ω_1, and AT be a tangent of ω_1. Line AB intersects ω_1 at C, and $\triangle ACT$ has circumcircle ω_2. Points P and S exist such that PA and PC are tangent to ω_2 and $SB = BT = 20$. Given that $AT = 15$, the length of PS can be written as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers, b is squarefree, and $\gcd(a, b) = 1$. Find $a + b + c$.

______________ 33. **[26]** There are a hundred students in math team. Each pair of students are either mutually *friends* or mutually *enemies*. It is given that if any three students are chosen, then they are not all mutually friends. The maximum possible number of ways to choose four students such that it is possible to label them A, B, C, and D such that A and B are friends, B and C are friends, C and D are friends, and D and A are friends can be expressed as n^4. Find n.

4th Annual Fall Lexington Math Tournament - Guts Round - Part 12

Team Name: ______________________________

______________ 34. **[30]** Let $\{p_i\}$ be the prime numbers, such that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$ For each i, let q_i be the nearest perfect square to p_i. Estimate $\sum_{i=1}^{2021} |p_i - q_i|$. If the correct answer is A and your answer is E, your score will be $\left\lfloor 30 \cdot \max\left(0, 1 - 5 \cdot \left|\log_{10} \frac{A}{E}\right|\right)\right\rfloor$.

______________ 35. **[30]** Estimate the number of digits of $(2021!)^{2021}$. If the correct answer is A and your answer is E, your score will be $\left\lfloor 15 \cdot \max\left(0, 2 - \left|\log_{10} \frac{A}{E}\right|\right)\right\rfloor$.

______________ 36. **[30]** Pick a positive integer between 1 and 1000, inclusive. If your answer is E and a quarter of the mean of all the responses to this problem is A, your score will be

$$\lfloor \max(0, 30 - |A - E|, 2 - |E - 1000|) \rfloor .$$

Note that if you pick 1000, you will automatically get 2 points.
