Team Round

Lexington High School

March 23, 2019

1. David runs at 3 times the speed of Alice. If Alice runs 2 miles in 30 minutes, determine how many minutes it takes for David to run a mile.

2. Al has 2019 red jelly beans. Bob has 2018 green jelly beans. Carl has x blue jelly beans. The minimum number of jelly beans that must be drawn in order to guarantee 2 jelly beans of each color is 4041. Compute x.

3. Find the 7-digit palindrome which is divisible by 7 and whose first three digits are all 2.

4. Determine the number of ways to put 5 indistinguishable balls in 6 distinguishable boxes.

5. A certain reduced fraction $\frac{a}{b}$ (with $a, b > 1$) has the property that when 2 is subtracted from the numerator and added to the denominator, the resulting fraction has $\frac{1}{6}$ of its original value. Find this fraction.

6. Find the smallest positive integer n such that $|\tau(n+1) - \tau(n)| = 7$. Here, $\tau(n)$ denotes the number of divisors of n.

7. Let $\triangle ABC$ be the triangle such that $AB = 3$, $AC = 6$ and $\angle BAC = 120^\circ$. Let D be the point on BC such that AD bisect $\angle BAC$. Compute the length of AD.

8. 26 points are evenly spaced around a circle and are labeled A through Z in alphabetical order. Triangle $\triangle LMT$ is drawn. Three more points, each distinct from L, M, and T, are chosen to form a second triangle. Compute the probability that the two triangles do not overlap.

9. Given the three equations

$$a + b + c = 0$$
$$a^2 + b^2 + c^2 = 2$$
$$a^3 + b^3 + c^3 = 19$$

   find abc.

10. Circle ω is inscribed in convex quadrilateral $ABCD$ and tangent to AB and CD at P and Q, respectively. Given that $AP = 175, BP = 147, CQ = 75$, and $AB \parallel CD$, find the length of DQ.

11. Let p be a prime and m be a positive integer such that $157p = m^4 + 2m^3 + m^2 + 3$. Find the ordered pair (p, m).

12. Find the number of possible functions $f : \{-2, -1, 0, 1, 2\} \to \{-2, -1, 0, 1, 2\}$ that satisfy the following conditions.
   (1) $f(x) \neq f(y)$ when $x \neq y$ (2) There exists some x such that $f(x)^2 = x^2$

13. Let p be a prime number such that there exists positive integer n such that

$$41pn - 42p^2 = n^3.$$

   Find the sum of all possible values of p.

14. An equilateral triangle with side length 1 is rotated 60 degrees around its center. Compute the area of the region swept out by the interior of the triangle.

15. Let $\sigma(n)$ denote the number of positive integer divisors of n. Find the sum of all n that satisfy the equation $\sigma(n) = \frac{n}{3}$.

16. Let C be the set of points $\{a, b, c\} \in \mathbb{Z}$ for $0 \le a, b, c \le 10$. Alice starts at $(0, 0, 0)$. Every second she randomly moves to one of the other points in C that is on one of the lines parallel to the x, y, and z axes through the point she is currently at, each point with equal probability. Determine the expected number of seconds it will take her to reach $(10, 10, 10)$.

17. ($\star$) Find the maximum possible value of
$$abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3$$
where a, b, c are real such that $a+b+c=0$.

18. Circle ω with radius 6 is inscribed within quadrilateral $ABCD$. ω is tangent to AB, BC, CD, and DA at E, F, G, and H respectively. If $AE=3$, $BF=4$ and $CG=5$, find the length of DH.

19. Find the maximum integer p less than 1000 for which there exists a positive integer q such that the cubic equation
$$x^3 - px^2 + qx - (p^2 - 4q + 4) = 0$$
has three roots which are all positive integers.

20. ($\star$) Let $\triangle ABC$ be the triangle such that $\angle ABC = 60^\circ$, $\angle ACB = 20^\circ$. Let P be the point such that CP bisects $\angle ACB$ and $\angle PAC = 30^\circ$. Find $\angle PBC$.