# Individual Round

Lexington High School

March 23, 2019

1. Compute $2020 \cdot \left(2^{(0\cdot 1)} + 9 - \frac{(20^1)}{8}\right)$.

2. Nathan has five distinct shirts, three distinct pairs of pants, and four distinct pairs of shoes. If an “outfit” has a shirt, pair of pants, and a pair of shoes, how many distinct outfits can Nathan make?

3. Let $ABCD$ be a rhombus such that $\triangle ABD$ and $\triangle BCD$ are equilateral triangles. Find the angle measure of $\angle ACD$ in degrees.

4. Find the units digit of $2019^{2019}$.

5. Determine the number of ways to color the four vertices of a square red, white, or blue if two colorings that can be turned into each other by rotations and reflections are considered the same.

6. Kathy rolls two fair dice numbered from 1 to 6. At least one of them comes up as a 4 or 5. Compute the probability that the sum of the numbers of the two dice is at least 10.

7. Find the number of ordered pairs of positive integers $(x, y)$ such that $20x + 19y = 2019$.

8. Let $p$ be a prime number such that both $2p - 1$ and $10p - 1$ are prime numbers. Find the sum of all possible values of $p$.

9. In a square $ABCD$ with side length 10, let $E$ be the intersection of $AC$ and $BD$. There is a circle inscribed in triangle $ABE$ with radius $r$ and a circle circumscribed around triangle $ABE$ with radius $R$. Compute $R - r$.

10. The fraction $\frac{13}{37\cdot 77}$ can be written as a repeating decimal $0.\overline{a_1 a_2 ... a_{n-1} a_n}$ with n digits in its shortest repeating decimal representation. Find $a_1 + a_2 + \ldots + a_{n-1} + a_n$.

11. Let point $E$ be the midpoint of segment $AB$ of length 12. Linda the ant is sitting at $A$. If there is a circle $O$ of radius 3 centered at $E$, compute the length of the shortest path Linda can take from $A$ to $B$ if she can’t cross the circumference of $O$.

12. Euhan and Minjune are playing tennis. The first one to reach 25 points wins. Every point ends with Euhan calling the ball in or out. If the ball is called in, Minjune receives a point. If the ball is called out, Euhan receives a point. Euhan always makes the right call when the ball is out. However, he has a $\frac{3}{4}$ chance of making the right call when the ball is in, meaning that he has a $\frac{1}{4}$ chance of calling a ball out when it is in. The probability that the ball is in is equal to the probability that the ball is out. If Euhan won, determine the expected number of wrong calls made by Euhan.

13. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ which contain four consecutive numbers.

14. Ezra and Richard are playing a game which consists of a series of rounds. In each round, one of either Ezra or Richard receives a point. When one of either Ezra or Richard has three more points than the other, he is declared the winner. Find the number of games which last eleven rounds. Two games are considered distinct if there exists a round in which the two games had different outcomes.

15. There are 10 distinct subway lines in Boston, each of which consists of a path of stations. Using any 9 lines, any pair of stations are connected. However, among any 8 lines there exists a pair of stations that cannot be reached from one another. It happens that the number of stations is minimized so this property is satisfied. What is the average number of stations that each line passes through?

16. There exist positive integers $k$ and $3 \nmid m$ for which

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{53} - \frac{1}{54} + \frac{1}{55} = \frac{3^k \times m}{28 \times 29 \times \cdots \times 54 \times 55}.$$

Find the value $k$.

17. Geronimo the giraffe is removing pellets from a box without replacement. There are 5 red pellets, 10 blue pellets, and 15 white pellets. Determine the probability that all of the red pellets are removed before all the blue pellets and before all of the white pellets are removed.

18. Find the remainder when

$$70!\left(\frac{1}{4 \times 67} + \frac{1}{5 \times 66} + \cdots + \frac{1}{66 \times 5} + \frac{1}{67 \times 4}\right)$$

is divided by 71.

19. Let $A_1A_2\ldots A_{12}$ be the regular dodecagon. Let $X$ be the intersection of $A_1A_2$ and $A_5A_{11}$. Given that $XA_2 \cdot A_1A_2 = 10$, find the area of dodecagon.

20. Evaluate the following infinite series:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n\sec^2 m - m\tan^2 n}{3^{m+n}(m+n)}.$$