Individual Round

Lexington High School

December 8, 2018

1. Find the area of a right triangle with legs of lengths 20 and 18.
2. How many 4-digit numbers (without leading zeros) contain only $2, 0, 1, 8$ as digits? Digits *can* be used more than once.
3. A rectangle has perimeter 24. Compute the largest possible area of the rectangle.
4. Find the smallest positive integer with 12 positive factors, including one and itself.
5. Sammy can buy 3 pencils and 6 shoes for 9 dollars, and Ben can buy 4 pencils and 4 shoes for 10 dollars at the same store. How much more money does a pencil cost than a shoe?
6. What is the radius of the circle inscribed in a right triangle with legs of length 3 and 4?
7. Find the angle between the minute and hour hands of a clock at $12:30$.
8. Three distinct numbers are selected at random from the set $\{1, 2, 3, \cdots, 101\}$. Find the probability that 20 and 18 are two of those numbers.
9. If it takes 6 builders 4 days to build 6 houses, find the number of houses 8 builders can build in 9 days.
10. A six sided die is rolled three times. Find the probability that each consecutive roll is less than the roll before it.
11. Find the positive integer n so that $\frac{8-6\sqrt{n}}{n}$ is the reciprocal of $\frac{80+6\sqrt{n}}{n}$.
12. Find the number of all positive integers less than 511 whose binary representations differ from that of 511 in exactly two places.
13. Find the largest number of diagonals that can be drawn within a regular 2018-gon so that no two intersect.
14. Let a and b be positive real numbers with $a > b$ such that $ab = a + b = 2018$. Find $\lfloor 1000a \rfloor$. Here $\lfloor x \rfloor$ is equal to the greatest integer less than or equal to x.
15. Let r_1 and r_2 be the roots of $x^2 + 4x + 5 = 0$. Find $r_1^2 + r_2^2$.
16. Let $\triangle ABC$ with $AB = 5, BC = 4, CA = 3$ be inscribed in a circle Ω. Let the tangent to Ω at A intersect BC at D and let the tangent to Ω at B intersect AC at E. Let AB intersect DE at F. Find the length BF.
17. A standard 6-sided die and a 4-sided die numbered 1, 2, 3, and 4 are rolled and summed. What is the probability that the sum is 5?
18. Let A and B be the points $(2,0)$ and $(4,1)$ respectively. The point P is on the line $y = 2x + 1$ such that $AP + BP$ is minimized. Find the coordinates of P.
19. Rectangle $ABCD$ has points E and F on sides AB and BC, respectively. Given that $\frac{AE}{BE} = \frac{BF}{FC} = \frac{1}{2}$, $\angle ADE = 30°$, and $[DEF] = 25$, find the area of rectangle $ABCD$.
20. Find the sum of the coefficients in the expansion of $(x^2 - x + 1)^{2018}$.
21. If p, q and r are primes with $pqr = 19(p + q + r)$, find $p + q + r$.
22. Let $\triangle ABC$ be the triangle such that $\angle B$ is acute and $AB < AC$. Let D be the foot of altitude from A to BC and F be the foot of altitude from E, the midpoint of BC, to AB. If $AD = 16, BD = 12, AF = 5$, find the value of AC^2.
23. Let a, b, c be positive real numbers such that

(i) $c > a$

(ii) $10c = 7a + 4b + 2024$

(iii) $2024 = \frac{(a+c)^2}{a} + \frac{(c-a)^2}{b}$.

Find $a + b + c$.

24. Let $f^1(x) = x^2 - 2x + 2$, and for $n > 1$ define $f^n(x) = f(f^{n-1}(x))$. Find the greatest prime factor of $f^{2018}(2019) - 1$.

25. Let I be the incenter of $\triangle ABC$ and D be the intersection of line that passes through I that is perpendicular to AI and BC. If $AB = 60, CA = 120$, and $CD = 100$, find the length of BC.