5th Annual Lexington Math Tournament Team Round

April 12, 2014

Name ______________________________ Team ______________________________

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1 Potpourri[80]

1. Let $A\%B = B^A - B - A + 1$. How many digits are in the number $1\%(3\%(3\%7))$?

2. Three circles, of radii 1, 2, and 3 are all externally tangent to each other. A fourth circle is drawn which passes through the centers of those three circles. What is the radius of this larger circle?

3. Express $\frac{1}{3}$ in base 2 as a binary number. (Which, similar to how demical numbers have a decimal point, has a "binary point".)

4. Isosceles trapezoid $ABCD$ with AB parallel to CD is constructed such that $DB = DC$. If $AD = 20$, $AB = 14$, and P is the point on AD such that $BP + CP$ is minimized, what is AP/DP?

5. Let $f(x) = \frac{5x-6}{x-2}$. Define an infinite sequence of numbers $a_0, a_1, a_2 \ . \ . \ .$ such that $a_{i+1} = f(a_i)$ and a_i is always an integer. What are all the possible values for a_{2014}?

6. $MATH$ and $TEAM$ are two parallelograms. If the lengths of MH and AE are 13 and 15, and distance from AM to T is 12, find the perimeter of $AMHE$.

7. How many integers less than 1000 are there such that $n^n + n$ is divisible by 5?

8. 10 coins with probabilities of 1, 1/2, 1/3 $\ldots$, 1/10 of coming up heads are flipped. What is the probability that an odd number of them come up heads?

9. An infinite number of coins with probabilities of 1/4, 1/9, 1/16, $\ldots$ of coming up heads are all flipped. What is the probability that exactly 1 of them comes up heads?

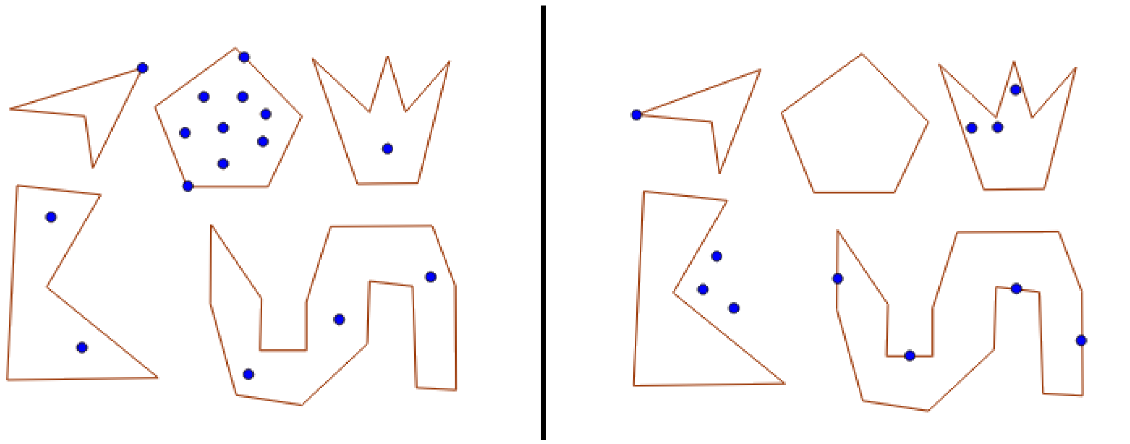
10. Quadrilateral $ABCD$ has side lengths $AB = 10, BC = 11, and CD = 13$. Circles O_1 and O_2 are inscribed in triangles ABD and BDC. If they are both tangent to BD at the same point E, what is the length of DA?

2 Long Answer[120]

This section deals with placing enough guards in an art gallery to fully guard all of the space inside. The problems will all use the following definitions:

An *art gallery* is defined to be a polygon in the plane. An art gallery includes the interior region as well as the boundary segments—*the walls*. We let G denote an arbitrary art gallery and write G_w for an art gallery with w walls. Let p be any point in an art gallery. The point q is visible to a *guard* positioned at p provided the line segment joining p and q does not exit the gallery. (We also assume that every point is visible to itself.) The segment represents the sight line of a guard. A set of guards protects an art gallery provided every point in the gallery is visible to at least one guard. Note that a guard at a corner protects the two adjacent walls.

For example, the galleries on the left are all protected, however, the ones on the right are not:



Let $M(w)$ be the minimum amount of guards that can necesarily guard all art galleries with w walls for $w \geq 3$

Problems 2, 3, 6, 7, 8, and 9 require proofs. Problems 1, 4, and 5 solely require a drawing of a guarded art gallery. (Copy the gallery for 1 as best you can, along with the amount of guards and an example way to guard it.)

1. (8) Find the smallest amount of guards needed to guard a regular star?
2. (10) Prove that $M(4) = 1$.
3. (12) Prove that every convex art gallery can be guarded with 1 guard.
4. (14) Draw an example of a art gallery, with the smallest possible amount of walls w such that $M(w) = 6$.
5. (16) Draw an example of a art gallery, with the smallest possible amount of walls, that can be uniquely guarded with 2 guards (In other words, only one possible positioning of those guards will guard the entire gallery).

   The LHS captains have come up with ideas on how to guard an art gallery. For each one, prove that they are right, or find a counter example.

   (*Hint:* Two of them are right and two of them are wrong)
6. (12) Zach thinks that every art gallery can be guarded by putting a guard at each corner.
7. (14) Shohini thinks that every art gallery can be guarded by putting a guard at the center of each wall.
8. (16) Rohil thinks that every art gallery with $2n$ walls can be guarded by at n guards by coloring the vertices of the gallery alternately black and white, and then putting guards at either the white vertices or the black vertices will always work.
9. (18) Noah thinks that every art gallery can be guarded by putting a guard at each corner with angle ≥ 180 degrees, unless it is a convex polygon.