4th Annual Lexington Mathematical Tournament
Theme Round

Solutions

1 Apple Pi and Other Desserts

1. Answer: $\boxed{73}$ If Surya sells 0 cookies, then the number of cookies was reduced by 36 cookies, so the price must have been increased by 4 cents, 12 times. Thus, the price must be $25+4\cdot 12 = 73$.

2. Answer: $\boxed{49}$ Let b be the number of sorbets, g be the number of gelatos, and d be the number of sundaes. We set up the system of equations

$$3b + 4g = 144$$
$$6g + 9b = 117$$
$$7b + 1d = 229.$$

   We can always substitute variables and work our way throughout, but we note that if we add the 3 equations together, we have $10b + 10g + 10d = 490$. Thus, $b + g + d = 49$, which is the price of buying one of each ice cream.

3. Answer: $\boxed{15}$ If a number has 4 factors, then it must either be in the form p^3 or pq for primes p and q. In the first case, there is only one number that is even, namely, $2^3 = 8$. In the second case, suppose $p = 2$, since the number must be divisible by 2. Now we look for all the numbers in the form of $2q$ that are less than 99. Thus, q can be 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 - the 14 primes less than or equal to 49. Thus, there are $14 + 1 = 15$ *delicious* numbers.

4. Answer: $\boxed{5/8}$ There are 2^5 different ways for Charlie to decorate the pies. In order to make the number of scoops on the sixth pie one more or one less than the number of scoops on the first pie, the last pie must have either 9 or 11 scoops of whipped cream. Therefore, he must increase the number of scoops either 2 or 3 times out of a total of 5 times he changes the number of scoops and decrease the number of scoops 3 or 2 times, respectively. As a result, there are $\binom{5}{2} + \binom{5}{3} = 20$ ways to have 9 or 11 scoops of whipped cream on the sixth pie. For each pie, he can either increase or decrease the number of scoops, so there are $2^5 = 32$ ways to decorate in total. The probability is $20/32 = 5/8$.

5. Answer: $\boxed{496}$ Hao can only drink all of the lemonade when he drinks the whole juice box, because there will always be some lemonade left over if he drinks any amount less than 32 ounces. Thus, he drinks $1+2+3+\cdots+32 = \frac{32\cdot 33}{2} = 528$ ounces of juice. Of those, 32 ounces were lemonade because he drank all of the original amount of lemonade and no lemonade was added. Therefore, Hao drank $528 - 32 = 496$ ounces of Hater-ade. In other words, Hao's hatin' reeeeeal hard.

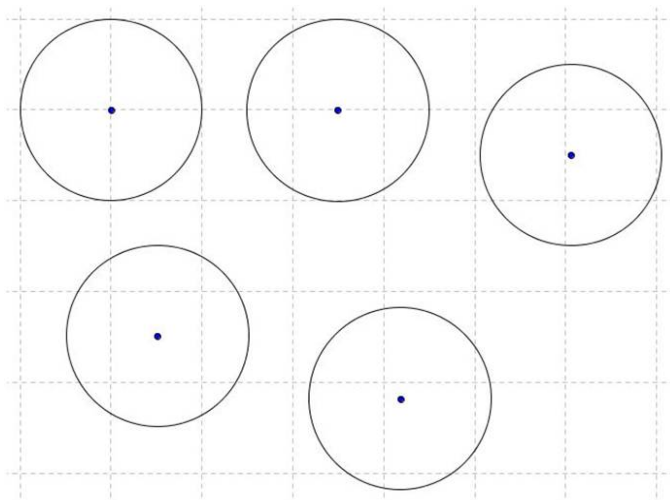
2 Game Theory

6. Answer: $\boxed{3}$ To get an 18, all 3 dice must roll a 6. This has probability $q = (\frac{1}{6})^3$. To get a 17, 2 dice must roll a 6 and 1 die must roll a 5. This has probability $p = 3(\frac{1}{6})^6$ since any of the 3 dice can roll the 5. Thus, $\frac{p}{q} = 3$.

7. Answer: $\boxed{1254}$ Using Bill's and Ben's hints, we can make a list of all possible numbers: 1111, 112, 121, 211, 13, 22, 31, 4. We factor the numbers and see if there is a single triplet of numbers that all share the same prime factor. We can see that 1111, 121, and 22 all have the same factor of 11. Thus, the sum is $1111 + 121 + 22 = 1254$.

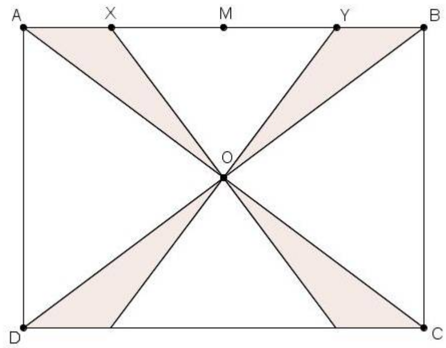
8. Answer: $\boxed{4}$ If we call the top left square black, then there are 4 whites and 5 blacks. Sara can achieve the task by choosing 4 squares - the 4 white squares. After choosing the 4 white squares, it turns out that all squares are black. To check that choosing 3 does not work, we note that the order in which we pick squares does not matter and that choosing a square twice is the same as if it were never chosen. Using these simplifying assumptions, we can quickly check all possibilities with 3 or fewer choices and find that none of them work. Thus, 4 is the minimum.

9. Answer: $\boxed{5}$ To cover the least number of squares, the checker must be placed a vertex of a square, where it can cover 4 squares. From there, we move it a little bit to the left so it will cover 2 more squares for a total of 6 squares. If the checker starts at a midpoint of a side of a square, we can move it a little in the direction perpendicular to the edge, allowing it to cover 7 squares. We can cover 9 squares by putting the checker in the middle of a square.



Now we try to cover 8 squares. Let's call the vertices of a square A, B, C, and D. Starting from the center of the square, the checker can move towards A until it is a little more than 1 from C. The center of the checker is still less than 1 from B and D because the entire diagonal from A to C is within 1 from B or D (this can be shown by drawing circles of radius 1 centered around B and D). Since the center of the checker is less than 1 from B or D, the checker covers 8 squares as shown. To show that the checker cannot cover 5 squares, we note that from the previous analysis, if a checker lies on a vertex or side of a square, it will cover 4 or 6 squares. Suppose the center of the circle O lies inside square ABCD. By drawing circles of radius 1 centered around B and D, it is evident that OB or OD is less than 1. Suppose $OB < 1$. Then, the checker piece will overlap with part of the interior of the square that shares only vertex B with $ABCD$. Similarly, one of OA or OC will be less than 1, and the checker piece will overlap that corresponding square. Thus, the checker piece must cover at least 6 squares if its center is in the interior of a square, so exactly 5 squares cannot be covered. Therefore, the checker can cover, 4, 6, 7, 8, 9 squares, and we clearly cannot cover more than 9 squares. The number of possible values of 5.

10. Answer: $\boxed{\dfrac{7}{32}}$ Label the rectangular room $ABCD$ with center O as shown. The lasers can only be fired at the $\overline{AB}$ or $\overline{CD}$ wall, since $\angle AOD$ and $\angle BOC$ are both acute. Let X and Y lie on $\overline{AB}$ such that $\angle AOY$ and $\angle BOX$ are both right angles. We wish to find the areas of $\triangle AOX$ and $\triangle BOY$, the regions where the person can be hit closer to side $\overline{AB}$.

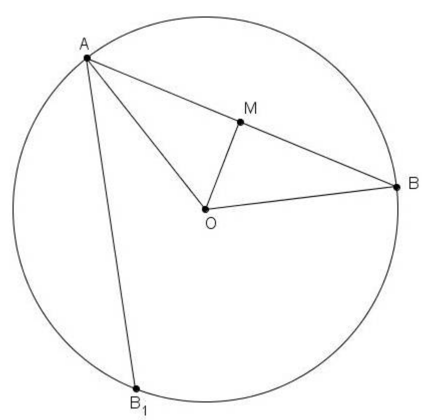


Call M the midpoint of $\overline{AB}$. Then, triangles AOM and YOM are similar, so $\frac{AM}{OM} = \frac{OM}{MY} \Rightarrow \frac{4}{3} = \frac{3}{MY}$, so $MY = \frac{9}{4}$ and $YX = 2MY = \frac{9}{2}$. Since triangles AOB and XOY share the same base, $[XOY]$ is $\frac{\frac{9}{2}}{8} = \frac{9}{16}[AOB]$. Therefore, the areas of triangles AOX and BOY make up $1 - \frac{9}{16} = \frac{7}{16}$ the area of triangle AOB. Triangle AOB is $\frac{1}{4}$ the area of $ABCD$, so these two regions are $\frac{7}{16} \cdot \frac{1}{4} = \frac{7}{64}$ the area of $ABCD$. The same region is on the other side of the O, so the probability is $\frac{7}{64} \cdot 2 = \frac{7}{32}$.

3 Dream Jobs

11. Answer: $\boxed{1.50}$ Let the price of one apple be x. Including tax, Frank pays $5x(1.05)$, which is known to be 2.50. So, $x(1.05) = .50$. George buys 3 apples so he pays $3x(1.05) = 3(.50) = 1.50$.

12. Answer: $\boxed{\frac{1}{3}}$ Let O be the center of the circle. When arc AB is 120 degrees, by letting M be the midpoint of $\overline{AB}$ and considering the 30-60-90 triangle AOM, we see that $AB = 4\sqrt{3}$. We can choose A to be anywhere, and then arc AB has to be more than 120 degrees in order to make $AB > 4\sqrt{3}$. This means B can be on a 120 degree arc from B to B_1, giving an answer of $\frac{120}{360} = \frac{1}{3}$.



13. Answer: $\boxed{30}$ Noah runs a lap every 10/3 minute while Jonah runs a lap every 15/4 minute. Each minute, Noah runs 3/10 lap while Jonah runs 4/15 lap. Therefore after every minute, the two are separated a distance of $3/10 - 4/15 = 1/30$ lap. It must take them 30 minutes in order for the distance between the two to be one lap, which is also when they are next to each other.

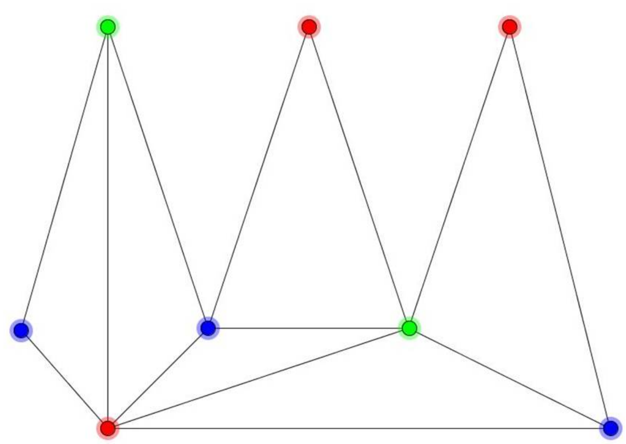
14. Answer: $\boxed{265}$ We break the language into mini cycles such that only letters from each cycle replace each other. Note that cycles cannot have size 1 since a different letter must replace a

given letter. Case 1: cycles of size 4, 2. There are $\binom{6}{2} = 15$ ways to split the letters. For a cycle with size 4, there are $3! = 6$ ways to create the replacement sequence. (The first letter has 3 letters that can replace it. The second has 2 and so on). There is 1 way to create the replacement sequence in a cycle with size 2. The total number of secret languages that follow this case is $15 \cdot 6 = 90$. Case 2: cycles of size 3, 3. There are $\binom{6}{3} = 20$ ways to split the letters but since both cycles have 3 elements, we must divide by 2 to correct for overcounting. For a cycle with size 3, there are 2! ways to create the replacement sequence (same reasoning as before). The total number of secret languages that follow this case is $(20/2) \cdot 2! \cdot 2! = 40$. Case 3: cycles of size 2, 2, 2. There are $\binom{6}{2}\binom{4}{2} = 90$ ways to split the letters but since all cycles have 2 elements, we must divide by 3! to correct overcounting. The total number of secret languages that follow this case is $90/3! = 15$. Case 4: cycle of size 6. For a cycle with size 6, there are 5! ways to create the replacement sequence (same reasoning as before). The total number of secret languages that follow this case is $5! = 120$. The total number of secret languages is $90 + 40 + 15 + 120 = 265$.

Note: This is a standard derangement problem. If the number of derangements of n objects is D_n, then D_n satisfies the recursion $D_n = (n-1)(D_{n-1} + D_{n-2})$. Alternatively, it can be directly computed as

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

15. Answer: $\boxed{2}$ Consider a simpler case: the room is a triangle. Obviously, we can place the guardian at any vertex and the task is complete. If the room were a quadrilateral, we can split it into 2 triangles which share a side. A guard placed at either of the two shared vertices would be able to complete the task since they can see all of both triangles. We extend this idea of placing guardians at vertices to a room in the shape of an octagon. For any given octagon, we can break it into triangles. We color the vertices of the octagon such that we only use 3 colors and every triangle has three vertices with different colors. Choosing a triangle, we can color the vertices any way we want. For any triangle that shares a side with the first triangle, we color the vertex that isn't shared with first triangle, and there is exactly one way to do this. We continue this procedure until all the vertices are colored. If we pick a guardian to stand at all vertices of a given color, all areas are guaranteed to be seen by at least one guardian since every triangle has one vertex of that color. Given 8 vertices, there must be some color that is only used at most twice, otherwise there would be at least $3 \cdot 3 = 9$ vertices, at least three of each color. Thus we need at most two guardians. Now we must show that we cannot use 1 guardian. We can construct a room in the shape as shown, which obviously cannot be secured with 1 guardian. Note that the two guardians can be placed at the green vertices.



Note: The coloring argument can be made more rigorous with mathematical induction.