4th Annual Lexington Mathematical Tournament
Team Round

Solutions

1 Potpourri

1. Answer: $\boxed{4}$ Alan's home clock is 6 minutes behind his watch. The school time is 2 minutes behind his watch. Thus Alan's home clock is 4 minutes behind the school clock.

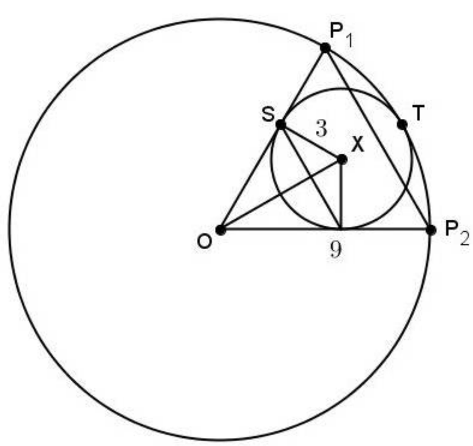
2. Answer: $\boxed{2012\frac{1}{2013}}$

$$\left(\frac{2012^{2012-2013}+2013}{2013}\right)\times 2012 = \frac{2012^{-1}+2013}{2013}\times 2012$$
$$= \frac{1+2013\times 2012}{2013}$$
$$= 2012\frac{1}{2013}.$$

3. Answer: $\boxed{2}$ Listing out the last digits of powers of 2, we have the sequence $2,4,8,6,2,\ldots$. Thus, the last digits of powers of two repeat in cycles of 4. It suffices to look for the remainder of the exponent, $3^{4^{5^{6^{7^{8^{9^{\cdot^{\cdot^{\cdot^{2013}}}}}}}}}}$, when divided by 4. When dividing by 4, we see that the powers of 3 leave remainders of $3,1,3,\ldots$. Thus, the remainders of the powers of 3 when divided by 4 repeat in cycles of 2, where odd powers leave a remainder of 3 and even powers leave a remainder of 1. Since 3^{4^x} is obviously an even power of 3, it leaves a remainder of 1 when divided by 4. Thus, the expression has a last digit of $2^1 = 2$.

4. Answer: $\boxed{36}$ $f(12) = f(2\times 6) = f(2)f(6) = f(2)f(2\times 3) = f(2)f(2)f(3) = 3\times 3\times 4 = 36$.

5. Answer: $\boxed{6}$ Let T be the point of tangency between the circles. Then, $OX = OT - XT = 9 - 3 = 6$. Let S be the tangency point of $\overrightarrow{OP_1}$ with circle X. Then, $\angle OSX = 90^\circ$, $OX = 6$, and $XS = 3$, so OSX is a 30-60-90 triangle with $\angle SOX = 30^\circ$. By symmetry, the other tangent line can be dealt with the same way, so $\angle P_1OP_2 = 60^\circ$ and $OP_1 = OP_2$. Therefore, OP_1P_2 is an equilateral triangle with $P_1P_2 = OP_1 = 6$.



6. Answer: $\boxed{23}$ Since this is a 24 hour clock, the 12 is across from the 24. For every hour, except 11 o'clock, there is only one time the hour and minute hands are opposite each other. We see that the time we would expect the hour and minute hands to be opposite when the hour hand is between 11 and 12 is actually when it is 12:00. Therefore, the hands will be in this formation $24 - 1 = 23$ times a day.

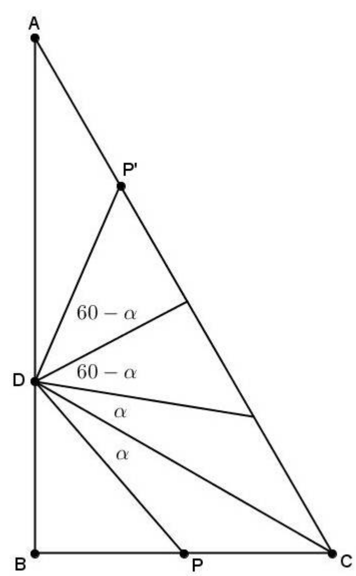
7. Answer: $\boxed{0}$ Before the candy is taken, there are $\binom{9}{3}\binom{6}{3}\binom{3}{3} = 1680$ ways to distribute the candy to the 3 kids. We do not divide by 3! since the kids are different. If there are only 8 pieces of candy that we distribute as evenly as possible, there are $3\left(\binom{8}{3}\binom{5}{3}\binom{2}{2}\right) = 1680$ ways to distribute the candy. Note we must multiply by 3 here since we must pick one of 3 children to only get 2 candies. The total number of ways to distribute the candies in both scenario is the same, so the difference is 0.

   Alternate solution: Consider an even distribution of the 9 candies to the 3 children. To go from this distribution to one nearly-even distribution of 8 candies to 3 children, we simply remove the candy that was taken by the mice. To go from one nearly-even distribution of 8 candies to 3 children to an even distribution of 9 candies to 3 children, we give the poor child with 2 candies the candy that was taken by the mice. Thus, there is a bijection between the two scenarios and the difference is 0.

8. Answer: $\boxed{3\sqrt{7}/4}$ Ronny's first fold is essentially folding angle C in half. Let point D be on $\overline{AB}$ such that $\overline{CD}$ is the crease of the fold. We see that Ronny's second fold is essentially folding $\angle ADC$ in half. Let $\angle PDC = \alpha$. After the first fold, P rotates 2α counterclockwise. After the next flip, P rotates another $2(60 - \alpha) = 120 - 2\alpha$ degrees counterclockwise. The net rotation is a 120 degree rotation. Therefore, $\angle P'DP = 120°$ with $DP = DP'$. We can draw the height from D to $\overline{PP'}$ to produce two 30-60-90 triangles. Therefore, $PP' = DP\sqrt{3}$. To find DP, we can use the Pythagorean theorem:

$$\begin{aligned} DB &= BC/\sqrt{3} = \sqrt{3}/2 \\ BP &= 3/4 \\ DP &= \sqrt{(DB)^2 + (BP)^2} = \sqrt{21}/4. \end{aligned}$$

   Finally, $PP' = \sqrt{3} \times \sqrt{21}/4 = 3\sqrt{7}/4$.



9. Answer: $\boxed{35}$ We proceed by casework.
Case 1: the integers have 1 digit. Only 1 and 2 have one digit in base 3.
Case 2: the integers have 2 digits. The bounds for base 3 integers are from 3 to $3^2 - 1 = 8$. The bounds for base 4 integers are from 4 to $4^2 - 1 = 15$. Therefore, only the 5 integers from 4 to 8 work.
Case 3: the integers have 3 digits. The bounds for base 3 integers are from $3^2 = 9$ to $3^3 - 1 = 26$. The bounds for the base 4 integers are from $4^2 = 16$ to $4^3 - 1 = 63$. Therefore, only the 11 integers from 16 to 26 work.
Case 4: the integers have 4 digits. The bounds for base 3 integers are from $3^3 = 27$ to $3^4 - 1 = 80$. The bounds for the base 4 integers are from $4^3 = 64$ to $4^4 - 1 = 255$. Therefore, only the 17 integers from 64 to 80 work.
Case 5 and beyond: The bounds for base 3 integers are from $3^4 = 81$ to $3^5 - 1 = 242$. The bounds for the base 4 integers are from $4^4 = 256$ to $4^5 - 1 = 1023$. For every $x > 4$, $4^x > 3^x - 1$, so there are no more possible integers.
There are $2 + 5 + 11 + 17 = 35$ total integers.

10. Answer: $\boxed{6\frac{15}{16}}$ We look at some small cases. If it were a 2×2 grid, the bug would be able to travel on every square no matter what, so the expected value would be 4. If it were a 2×3 grid, we consider the cases of when the bug moves south.
Case 1: bug moves south first. The bug must move east, and then the bug is working with a 2×2 grid, allowing it to cover 6 squares. This happens with a chance of 1/2. Case 2: the bug moves east then south. The bug can then move west, where it covers 4 squares, or it can move east, where it covers 5 squares. Both of these happen with a chance of $(1/2)^3 = 1/8$.
Case 3: the bug moves east two times and then south. The bug can then only move west and cover 6 squares. This happens with a chance of $(1/2)^2 = 1/4$. Thus for a 2×3 grid, the expected number of squares is $6(1/2) + 4(1/8) + 5(1/8) + 6(1/4) = 45/8$.

Continuing this for a 2×4 grid, we have:
Case 1: bug moves south first. The bug must move east, and then the bug is working with a 2×3 grid, allowing it to cover an expected $2 + 45/8$ squares. This happens with a chance of 1/2.
Case 2: the bug moves east then south. The bug can then move west, where it covers 4 squares, or it can move east, and then the bug is working with a 2×2 grid and it can cover an expected 7 squares. Both of these happen with a chance of $(1/2)^3 = 1/8$.
Case 3: the bug moves east two times and then south. The bug can then move west, where it covers 6 squares, or it can move east, where it covers 6 squares. Both of these happen with a chance of $(1/2)^4 = 1/16$.
Case 4: the bug moves east three times and then south. The bug can then only move west and cover 8 squares. This happens with a chance of $(1/2)^3 = 1/8$. Thus for a 2×4 grid, the expected number of squares is

$$(61/8)(1/2) + 4(1/8) + 7(1/8) + 6(1/16) + 6(1/16) + 8(1/8) = \frac{111}{16} = 6\frac{15}{16}.$$

2 Long Answer

2.1 Fibbonacci Numbers [35]

1) Plugging in $n = 2, 3, 4, 5$, we have $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8$. Thus,

$$\begin{aligned}
F_3^2 - F_1^2 &= 3 = F_4 \\
F_4^2 - F_2^2 &= 8 = F_6 \\
F_5^2 - F_3^2 &= 21 = F_8 \\
F_6^2 - F_4^2 &= 55 = F_{10}.
\end{aligned}$$

2) From the pattern observed in Part 1, we notice that $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$.

3) We will prove the result by induction. The base case $k = 1$ follows from the definition:

$$F_n = F_{n-1} + F_{n-2} = F_{n-1}F_2 + F_{n-2}F_1.$$

Suppose the result is true for $k = m$; that is,

$$F_n = F_{n-m}(F_{m+1}) + F_{n-m-1}(F_m).$$

By definition, $F_{n-m} = F_{n-m-2} + F_{n-m-1}$. Substituting this in, we have

$$\begin{aligned}
F_n &= (F_{n-m-2} + F_{n-m-1})(F_{m+1}) + F_{n-m-1}(F_m) \\
&= F_{n-m-2}(F_{m+1}) + F_{n-m-1}(F_{m+1}) + F_{n-m-1}(F_m) \\
&= F_{n-m-2}(F_{m+1}) + F_{n-m-1}(F_{m+1} + F_m).
\end{aligned}$$

By definition, $F_m + F_{m+1} = F_{m+2}$, so

$$F_{n-m-2}(F_{m+1}) + F_{n-m-1}(F_{m+1} + F_m) = F_{n-(m+1)-1}(F_{(m+1)}) + F_{n-(m+1)}(F_{(m+1)+1}).$$

This completes the induction since we conclude the result for $k = m + 1$ given the result for $k = m$.

4) The Fibonacci number in part 2 is F_{2n}. We use $k = n$ in part 3 and express F_{2n} as $F_nF_{n+1} + F_{n-1}F_n$, which has the 3 consecutive Fibonacci numbers F_{n-1}, F_n, and F_{n+1}

5) We factor the expression in part 4 as $F_nF_{n+1} + F_{n-1}F_n = F_n(F_{n-1} + F_{n+1})$. By definition, $F_{n+1} = F_n + F_{n-1}$, or $F_n = F_{n+1} - F_{n-1}$. Substituting, we have

$$F_n(F_{n-1} + F_{n+1}) = (F_{n+1} - F_{n-1})(F_{n-1} + F_{n+1}) = F_{n+1}^2 - F_{n-1}^2.$$

2.2 Big Numbers [50]

6) We move everything to the left side, so $a^2 + b^2 + c^2 + d^2 - abcd - 6 = 0$. In the form required, this is $a^2 + (-bcd)a + (b^2 + c^2 + d^2 - 6) = 0$. Thus, $u = -bcd$ and $v = b^2 + c^2 + d^2 - 6$.

7) For a general quadratic equation $\alpha x^2 + \beta x + \gamma = 0$, the sum of the solutions for x is $-\beta/\alpha$. (To prove this, either use the quadratic formula or factor the quadratic as $\alpha(x - r)(x - s)0$ and expand.) Plugging in $\alpha = 1$ and $\beta = -bcd$, the sum of the two solutions is equal to bcd.

   If a is a solution for some constant values of b, c, and d, then since the sum of the two solutions is bcd, the other solution is $bcd - a$. Thus, $(bcd - a, b, c, d)$ is another quadruple of integers satisfying the given equation.

8) The solution is $(a, b, c, d) = (1, 2, 3, 4)$, or any rearrangement of the numbers. If we wish to brute force to find it, we can narrow down the list of solutions to check by plugging $abcd = 24$ into the right hand side of the equation.

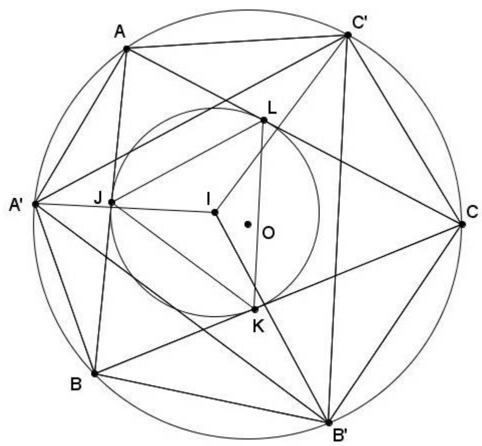
   Note: Any solution that satisfies the two equations given would be accepted here. However, negative values make the next problem rather difficult to do.

9) For any solution (a, b, c, d), without loss of generality, suppose a is the smallest value. Then, the other solution is $bcd - a$. Since a is the smallest value, $bcd - a \geq a^3 - a$. For all $a \geq 2$, $a^3 - a > a$, so one value in the solution increased. Furthermore, since this was originally the smallest value in the solution, the smallest value increases for each new solution generated. Thus, we just need to get a solution with $a, b, c, d \geq 2$. To do this, we take the solution $(a, b, c, d) = (1, 2, 3, 4)$ and get the other solution $(2 \cdot 3 \cdot 4 - 1, 2, 3, 4) = (23, 2, 3, 4)$. We iterate this process, and since the smallest value increases with each step because we choose to change the smallest value every time, eventually all of the values will be more than 2013.

Note 1: The fact that we operate on the smallest value is important in order to ensure that this process actually has the desired effect of increasing values each time. For example, if we started by operating on the 4 instead of the 1, we would get $(1, 2, 3, 4) \rightarrow (1, 2, 3, 2)$.

Note 2: This technique is known as *Vieta jumping* or *root flipping*, because we flipped roots to get some desired result. Generally, this is either used to show that a large solution exists, as in the case here, or to show that no solution exists, by supposing a solution exists, finding a smaller one given that solution, and reaching a contradiction because it is impossible to start from a positive integer and infinitely decrease it to a smaller positive integer.

2.3 Hexagon Area [45]



10) We first find the angles of triangle $A'B'C'$. It is clear that $m\angle C'A'B' = m\angle CA'B' + m\angle CA'C'$. Because $m\angle CA'C'$ and $m\angle CB'C'$ subtend the same arc, their angles are equal. Thus, $m\angle CA'C' = m\angle CB'C' = \frac{m\angle ABC}{2} = \frac{\beta}{2}$. Similarity, $m\angle CA'B' = m\angle CAB' = \frac{m\angle CAB}{2} = \frac{\alpha}{2}$. Therefore, $m\angle C'A'B' = \frac{\alpha+\beta}{2}$. By repeating this method for the other two angles, we get $m\angle A'B'C' = \frac{\beta+\gamma}{2}$ and $m\angle A'C'B' = \frac{\alpha+\gamma}{2}$.

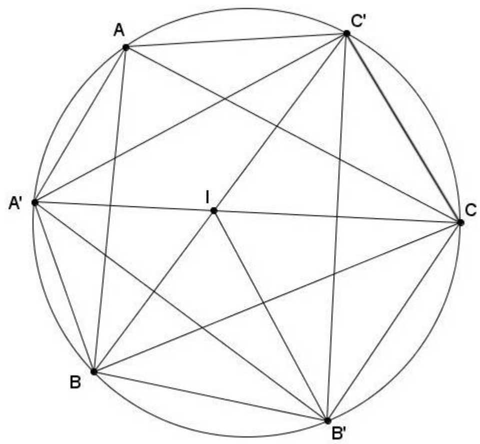
    Now we find the angles of triangle JKL. We have $m\angle KJL = 180 - m\angle AFL - m\angle BJK$. Since $\overline{BJ}$ and $\overline{BK}$ are tangents to the incircle and J and K lie on the circle, $BJ = BK$. This means that triangle BJK is isosceles and therefore, $m\angle BJK = \frac{180-\beta}{2}$. Similarly, we find that $m\angle AJL = \frac{180-\alpha}{2}$. Thus, $m\angle KJL = 180 - \frac{360-\alpha-\beta}{2} = \frac{\alpha+\beta}{2}$. By repeating this method for the other two angles, we find that $m\angle JLK = \frac{\alpha+\gamma}{2}$ and $m\angle JKL = \frac{\beta+\gamma}{2}$. By AAA similarity, we conclude that $A'B'C'$ is similar to triangle JKL.

11) We note that the angle bisector of the angles at A, B, and C all contain I as well as the midpoint of their respective subtended arcs. Since C' is the midpoint of arc AC, $m\angle AA'C' = m\angle CA'C' = m\angle IA'C'$. Since A' is the midpoint of arc AB, $m\angle AC'A' = m\angle BC'A' = m\angle IC'A'$. Since triangles $AA'C'$ and $IA'C'$ share the same side $\overline{A'C'}$, by ASA congruence,

triangles $AA'C'$ and $IA'C'$ are congruent. By symmetry, we know that $\triangle BA'B' \cong \triangle IA'B'$ and $\triangle CC'B' \cong \triangle IC'B'$. Thus, using $[\mathcal{F}]$ to denote the area of figure $\mathcal{F}$,

$$\begin{aligned}[AA'BB'CC'] &= [AA'C'] + [IA'C'] + [BA'B'] + [IA'B'] + [CC'B'] + [IC'B'] \\ &= 2([IA'C'] + [IA'B'] + [IC'B']) \\ &= 2[A'B'C'].\end{aligned}$$

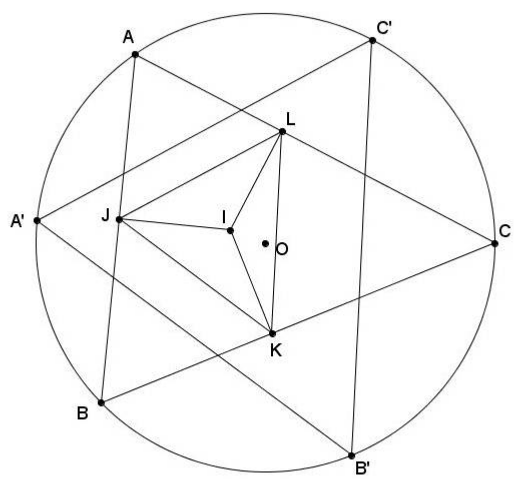
the area of hexagon $AA'BB'CC'$ is $AA'C' + IA'C' + BA'B' + IA'B' + CC'B' + IC'B' = 2 * (IA'C' + IA'B' + IC'B') = 2 * \triangle A'B'C'$.



12) Since JKL and $A'B'C'$ are similar triangles, the area of $A'B'C'$ is the area of JKL multiplied by the square of the ratio of their side lengths. Therefore,

$$[A'B'C'] = [JKL] \times \left(\frac{A'C'}{JL}\right)^2 = \frac{S}{r^2}.$$

In problem 11 we showed that $[AA'BB'CC'] = 2[A'B'C']$ so the area of the hexagon is $\boxed{\frac{2S}{r^2}}$.



13) From problem 12, let $JL/A'C' = r$. Since triangles JKL and $A'B'C'$ are similar, the ratio of their circumradii is also equal to r. We are given that the circumradius of triangle ABC is $\frac{65}{8}$, so we must find the circumradius of triangle JKL, which is also the inradius of triangle ABC. Call the inradius r. We see that $[ABC] = [IAB] + [IBC] + [IAC]$. Since the inradii of the triangle form right angles with the sides of the triangle, we have $[ABC] = \frac{AB \cdot r + BC \cdot r + AC \cdot r}{2} = \frac{r(AB+BC+AC)}{2} = 21r$. The area of triangle ABC is 84, which can be found by Heron's formula. Thus, $21r = 84$ so $r = 4$ and $JK/A'C' = \frac{4}{\frac{65}{8}} = \frac{32}{65}$. We are given that $S = \frac{1344}{65}$. From problem 12, the area of the hexagon is

$$\frac{2S}{r^2} = \frac{2 \cdot \frac{1344}{65}}{\left(\frac{32}{65}\right)^2} = \boxed{\frac{1365}{8}}.$$

Note: The length for the circumradius of triangle ABC by using the fact that the circumcenter is the intersection of the perpendicular bisectors of the sides and using the Pythagorean theorem, or simply by the formula $R = \frac{abc}{4[ABC]}$. The area of triangle JKL can be found by taking the area of triangle ABC and subtracting the areas of triangles AJL, BJK, and CKL. The area of these triangles can be found by comparing areas of triangles of different heights and same bases or vice versa.