# 4th Annual Lexington Mathematical Tournament
# Guts Round

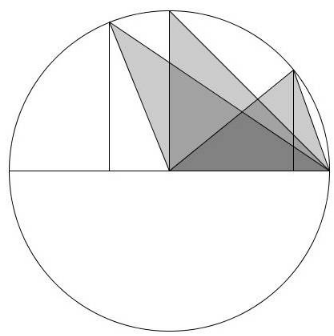
Solutions

1. Answer: $\boxed{9}$ The smallest power greater than 3 is 4, which is $2^2$. The greatest power less than 2013 is 1024, which is $2^{10}$. Thus there are $10 - 1 = 9$ powers of 2 between 3 and 2013.

2. Answer: $\boxed{-3}$ Let $x$ be the answer to the question. The number referred to in the beginning is the answer to the question, which is $x$. Thus, $x = 6 + 3x$, so $-2x = 6$ and $x = -3$.

3. Answer: $\boxed{69\pi}$ The exterior surface area is the sum of the area of the single base and the lateral surface area. The area of the base is $3^2\pi$ and the lateral surface area is $2 \cdot 3\pi \cdot 10 = 60\pi$, so the sum is $69\pi$.

4. Answer: $\boxed{18}$ Setting one side as the base, the other side can be pivoted around one of the endpoints (the shared endpoint) 360 degrees. To achieve the maximum area, the height must be maximal, which occurs when the sides are at a 90 degree angle. The height and base are both 6, so the area is $6 \cdot 6/2 = 18$.



5. Answer: $\boxed{5}$ We can just list the pairs of integers that are 5 apart with the smaller one less than 15 and the greater one greater than or equal to 10: (10,15); (11,16); (12,17); (13,18); (14,19). There are 5 possible dollar amounts Ethan can have.

6. Answer: $\boxed{2x}$ Squaring both sides of the given equation, we have $2013^2 = 2012^x \cdot 2012^x = 2012^{2x}$, so $a = 2x$.

7. Answer: $\boxed{7}$ Subtracting one from both sides, we have

$$\frac{1}{l + \frac{1}{m + \frac{1}{t}}} = \frac{36}{43}$$
$$l + \frac{1}{m + \frac{1}{t}} = \frac{43}{36}.$$

Since $l$, $m$, and $t$ are positive integers, the fraction is less than 1, so $l = 1$. Substituting, we get

$$\frac{1}{m + \frac{1}{t}} = \frac{7}{36}$$
$$m + \frac{1}{t} = \frac{36}{7}.$$

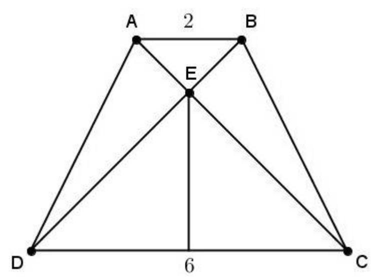
For the same reason, $m = 5$, and $1/t = 1/7 \Rightarrow t = 7$.

8. Answer: $\boxed{4}$ After the first year Jonathan has $1.2 \cdot 300,000 = 360,000$ in the bank. After the second year he has $1.2 \cdot 360,000 = 432,000$ in the bank. After the third year he has $1.2 \cdot 432,000 = 518,400$. After the fourth year he has over 600,000 dollars, so he can buy the land. Thus, he must wait 4 years.

9. Answer: $\boxed{288}$ After 2 hours, Arul swam 320 laps, so he swam 160 laps per hour. Ethan completed $160 + 32 = 192$ laps the first hour. In the second hour, he went half his original speed so he only completed $192/2 = 96$ laps. In total, Ethan swam $192 + 96 = 288$ laps.

10. Answer: $\boxed{24 - \dfrac{\pi}{2}}$ By the Pythagorean theorem, the other leg of the triangle is $\sqrt{10^2 - 6^2} = 8$. The area of the triangle is $6 \cdot 8/2 = 24$. The total sum of the angles in a triangle is 180 degrees so the area inside the triangle and a circle is $1^2\pi \cdot \frac{180}{360} = \pi/2$. The area inside the triangle and outside the circles is therefore $24 - \frac{pi}{2}$.

11. Answer: $\boxed{3}$ Since $\overline{AB} \parallel \overline{CD}$, $m\angle ABD = m\angle BDC$ and $m\angle BAC = m\angle ACD$, so triangles $ABE$ and $CDE$ are similar. Since $\frac{BE}{DE} = \frac{2}{6} = \frac{1}{3}$, the corresponding altitudes of $ABE$ and $CDE$ must be in ratio $\frac{1}{3}$. The altitude of $CDE$ from $E$ is the distance to $\overline{CD}$, which is 9, so the altitude of $ABE$ from $E$ is $\frac{9}{3} = 3$. The area of $ABE$ is $\frac{2 \cdot 3}{2} = 3$.




12. Answer: $\boxed{2}$ If the perimeter of the base is 54, the length and width must add up to 27. The product of the length and width must also be a factor of 144 since the height is an integer value. If one of the dimensions is $x$, $27 - x$ is the other dimension and $x(27 - x)$ divides 144. Testing all possible values of $x$ that divide 144 and are under 27, $x = 3$ and $x = 24$ are the only solutions, which correspond to the same prism. The height is therefore $144/(3 \cdot 24) = 2$.

13. Answer: $\boxed{-\dfrac{1}{3}}$ We can factor the first equation as $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, so

$$x^2 - xy + y^2 = \frac{x^3 + y^3}{x + y} = \frac{208}{4} = 52.$$

In addition, $(x + y)^2 = x^2 + 2xy + y^2 = 16$. Therefore,

$$3xy = (x^2 + 2xy + y^2) - (x^2 - xy + y^2) = 16 - 52 = -36,$$

so $xy = -12$. The sum of the reciprocals of $x$ and $y$ is

$$\frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy} = \frac{4}{-12} = -1/3.$$

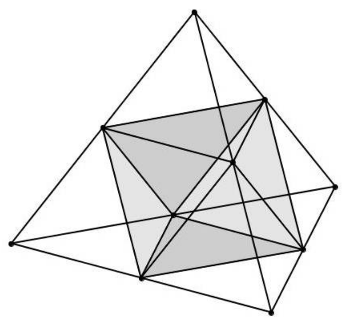
We can also note that $(x, y) = (6, -2)$ is a solution.

14. Answer: $\boxed{145}$ We first note that 7!, 8!, and 9! are all greater than 1000, so 7, 8, and 9 cannot be any of the digits. If 6 were a digit, then the integer must be at least 720 which would mean

it would have 7, 8, or 9 as a digit, which is impossible. If all the digits were less than or equal to 4, then the maximum integer would be $4! + 4! + 4! = 72$ which is not a 3 digit integer. Therefore, 5 must be a digit. There can either be 1 or 2 fives since 555 obviously does not work. If there were two 5s, there must also be a 2 in the hundreds place since $5! + 5! = 240$, but 255 does not work. Thus there is only one 5 which also means there is a 1 in the hundreds place, since $5! = 120$. Letting the remaining digit be 0, 1, 2, 3, or 4, the only integer that satisfy the conditions is 145.

15. Answer: $\boxed{7}$ Let each light have values 1 or 0, denoting it being on or off. Then, we can represent all possible combinations of the three lights with the coordinates $(x, y, z)$, where $x, y, z$ are 0 or 1, which happens to be a cube. We then look for the path that travels to every point using the minimum number of edges. This can obviously be done in 7 edge lengths.

16. Answer: $\boxed{26}$ There are 8 faces: 4 original faces and 4 faces created after chopping off 4 tetrahedrons. Every tetrahedron chopped off creates 3 edges so there are $3 \cdot 4 = 12$ edges. There are 6 vertices in total, three on one face and three on the parallel face. Alternatively, we can use the fact that $V + F - E = 2$ for a convex polyhedron, so $V = 2 + 12 - 8 = 6$. The total number of faces, vertices, and edges is $8 + 6 + 12 = 26$.

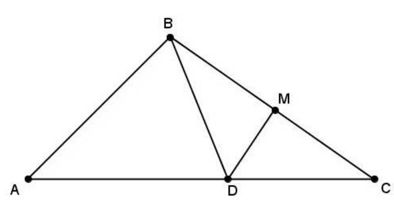


17. Answer: $\boxed{2011}$ We claim that for a convex $n$-gon, any triangulation has $n - 2$ triangles. We prove this by strong induction. For $n = 3$, this is obvious. Suppose the result is true for $n = 3, 4, \ldots, k$. For $n = k + 1$, consider a segment that divides the $(k + 1)$-gon into an $m$-gon and an $(k+3-m)$-gon. By the inductive hypothesis, this gives us $m-2+k-m+1 = (k+1)-2$ triangles, as desired.

    The conditions of the problem just state we are triangulating a regular 2013-gon. This gives us the answer $2013 - 2 = 2011$.

18. Answer: $\boxed{50}$ We know that $\frac{x^3+x-1000}{x^2+1} = x - \frac{1000}{x^2+1}$ must be an integer so $x^2 + 1$ must divide 1000. Listing some factors of 1000 from highest to lowest we see that $7^2 + 1 = 50$ is the largest possible value of $x$ that satisfies the conditions.

19. Answer: $\boxed{37}$ Call $M$ the midpoint of $\overline{BC}$. Since $BM = MC$, and $\overline{MT}$ is perpendicular to $\overline{BC}$, triangle $BTC$ is isosceles with $BT = CT$ and $m\angle TBC = m\angle TCB = 29°$. Furthermore, $m\angle BTC + m\angle TCB + m\angle TBC = 180°$, so $m\angle BTC = 180 - 29 - 29 = 122°$. Therefore, $m\angle ATB = 180 - m\angle BTC = 180 - 122 = 58°$ and $m\angle ABT = 87 - m\angle TBC = 87 - 29 = 58°$. Thus, triangle $ATB$ is isosceles with $AB = AT$, so $AC = AT + TC = AB + BT = 37$.



20. Answer: $\boxed{18}$ If $n$ is even, let $n = 2a$. The denominator of $f(2a)$ is $2 \cdot \frac{a(a+1)}{2} = a(a+1)$. The numerator of $f(2a)$ is $a(a+1) - a = a^2$. Thus, $f(2a) = a^2/(a^2 + a) = a/(a+1) = 1 - 1/(a+1)$. Then,

$$|1 - f(2a)| \le \frac{1}{10} \Rightarrow \frac{1}{a+1} \le \frac{1}{10} \Rightarrow a \ge 9.$$

This corresponds to a minimum value for $n$ of $2 \cdot 9 = 18$.

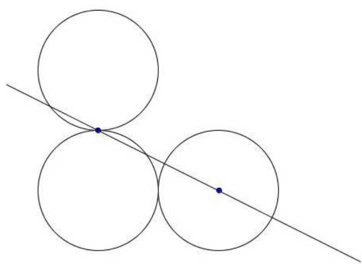
If $n$ is odd, let $n = 2a - 1$ with $a > 1$. The numerator of $f(2a-1)$ is the same as the numerator of $f(2a)$. The denominator, though is $a^2 + a - 2a = a^2 - a$. Thus, $f(2a-1) = a^2/(a^2 - a) = a/(a-1) = 1 + 1/(a-1)$. Then,

$$|1 - f(2a-1)| \le \frac{1}{10} \Rightarrow \frac{1}{a-1} \le \frac{1}{10} \Rightarrow a \ge 11.$$

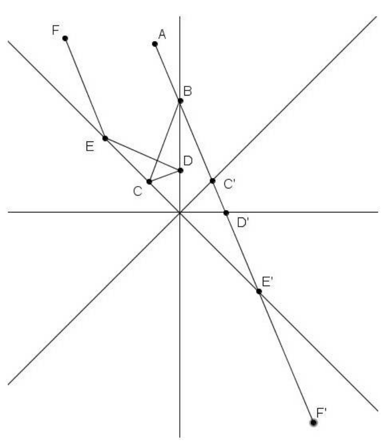
This corresponds to a minimum value for $n$ of $2 \cdot 11 - 1 = 21$.

Of these two cases, the minimum value for $n$ is 18.

21. Answer: $\boxed{y = -1/2x + 1}$ Consider the circles centered at (0,0) and (0,2). The line passes through (0,1), so this region is split evenly regardless of the slope of the line. Thus, the circle centered at (2,0) must be split evenly by the line. This means the line passes through the center, (2,0). This line has slope $-1/2$ and $y$-intercept 1, so its equation is $y = -1/2x + 1$.



22. Answer: $\boxed{4}$ When an object bounces into wall, the path of the ball can be represented by drawing the line of motion past the wall and reflecting that path over the wall. Therefore, we can continue to reflect the wall across each other until the ball cannot hit another wall. As shown in the diagram, the path of the ball denoted by $ABCDEF$ can be represented by the line $ABCDEF$. Since the ball moves in a line which has 180 degree angle, the ball can intersect the walls at most $180/45 = 4$ times.



23. Answer: $\boxed{\frac{25}{216}}$ Let $p$ be the probability the coin shows up heads. Based on Zachs game, the probability that he won without knowing that at least 3 heads showed up is $p^3(1-p)$. The probability that at least 3 heads showed up is $p^3(1-p)+p^4$. Therefore, the conditional probability given is

$$\frac{p^3(1-p)}{p^3(1-p)+p^4} = \frac{4}{9}.$$

Solving this equation, we have $9p^3(1-p) = 4(p^3(1-p)+p^4)$, so $9(1-p) = 4(1-p)+4p$ and $p = 5/6$. Therefore, the probability that Richard guesses correctly is

$$\binom{4}{2}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{25}{216}.$$

24. Answer: $\boxed{\frac{455}{144}}$ The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. Therefore, $S$ is the set with all integers with only factors of 2, 5, 7 or 13. Let an integer be in the form $2^a \cdot 5^b \cdot 7^c \cdot 13^d$. Then, its reciprocal would be $(1/2^a)(1/5^b)(1/7^c)(1/13^d)$. If we were to sum all numbers in this form over all $a$, $b$, $c$, $d$, this would be equivalent to

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots\right)\left(1+\frac{1}{7}+\frac{1}{7^2}+\cdots\right)\left(1+\frac{1}{13}+\frac{1}{13^2}+\cdots\right)$$

By choosing one term from each infinite sum, this product produces all possible values of $(1/2^a)(1/5^b)(1/7^c)(1/13^d)$. Using the fact that $1+r+r^2+\cdots = 1/(1-r)$ for $|r|<1$, this expression equals

$$\frac{1}{1/2}\frac{1}{4/5}\frac{1}{6/7}\frac{1}{12/13} = 2(5/4)(7/6)(13/12) = 455/144.$$

25. Answer: $\boxed{2646}$ We use casework based on whether 0 is a digit.

Case 1: There are no zeroes in the number. We choose 5 digits out of 9, and suppose the highest number is $x$. We can place $x$ in either the thousands, hundreds or tens digit. If it is placed in the thousands digit, we choose 1 of the remaining 4 numbers to be in the ten thousands digit and the rest will be fixed. If $x$ is in the hundreds digit, we choose 2 of the remaining 4 numbers to be in front and everything is fixed. If $x$ is in the tens digit, we choose a number to be the ones digit and everything is fixed. The total number of numbers in this case is

$$\binom{9}{5}\left(\binom{4}{1}+\binom{4}{2}+\binom{4}{1}\right) = 126 \cdot 14 = 1764.$$

Case 2: There is a zero in the number, which must be in the ones digit. We choose 4 numbers out of 9 and call the highest number $x$. We can place $x$ in either the thousands, hundreds or tens digit. If it is placed in the thousands digit, we choose 1 of the remaining 3 numbers to be in the ten thousands digit and the rest will be fixed. If $x$ is in the hundreds digit, we choose 2 of the remaining 3 numbers to be in front and everything is fixed. If $x$ is in the tens digit, everything is fixed. The total number is

$$\binom{9}{4}\left(\binom{3}{1}+\binom{3}{2}+1\right) = 126 \cdot 7 = 882.$$

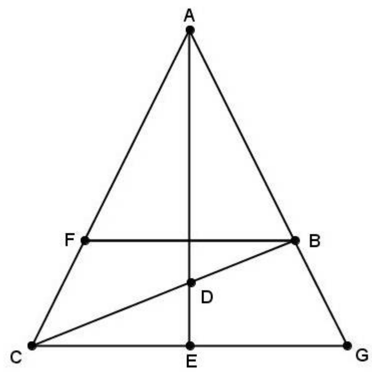
The total is $1764 + 882 = 2646$.

26. Answer: $\boxed{\frac{4}{5}}$ Let the perpendicular from $B$ to $\overline{AD}$ intersect $\overline{AC}$ at $F$, $\overline{BF}$ and $\overline{AD}$ intersect at $M$, and $\overline{AB}$ and $\overline{EC}$ intersect at $G$. Since $\overline{AE}$ is the angle bisector of $\angle BAC$ and $\overline{AE}$ is

perpendicular to $\overline{GC}$, triangle $GAC$ is isosceles with $GA = AC$. Since $\overline{BF}$ and $\overline{GC}$ are both perpendicular to $\overline{AE}$, they must be parallel and triangles $ABF$ and $AGC$ are similar. Thus,

$$\frac{AB}{AC} = \frac{4}{6} = \frac{2}{3},$$

so $\frac{AM}{AE} = \frac{2}{3}$. Let $AM = x$. Then, $AE = \frac{3x}{2}$ and $ME = AE - AM = \frac{x}{2}$.




We know that $m\angle MDB$ and $m\angle EDC$ must be equal as are $m\angle BMD$ and $m\angle DEC$ since they are both 90 degrees. Therefore, triangles $BMD$ and $CED$ are similar to each other. We know $\frac{BF}{GC} = \frac{4}{6}$ so

$$\frac{\frac{BF}{2}}{\frac{GC}{2}} = \frac{BM}{EC} = \frac{4}{6},$$

which means $\frac{MD}{DE} = \frac{4}{6} = \frac{2}{3}$ and $DE = \frac{3MD}{2}$. Since $MD + DE = ME = \frac{x}{2}$, we know

$$MD + \frac{3MD}{2} = \frac{x}{2} \Rightarrow MD = \frac{x}{5}.$$

Thus, $AD = AM + MD = x + \frac{x}{5} = \frac{6x}{5}$ and

$$\frac{AD}{AE} = \frac{\frac{6x}{5}}{\frac{3x}{2}} = \frac{4}{5}.$$

27. Answer: $\boxed{\pm 4\sqrt{6}}$ We notice that $(7 + 4\sqrt{3})(7 - 4\sqrt{3}) = 1$. Thus we let $a = (7 + 4\sqrt{3})^x$ and $b = (7 - 4\sqrt{3})^x$, so $a + b = 10$ and $ab = 1$. We would like $a - b$, so we have

$$(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab = 100 - 4 = 96,$$

so $a - b = \pm 4\sqrt{6}$.

28. Answer: $\boxed{1}$ The first equation can be written as $A + B = X - 8/5$. The second equation can be written as

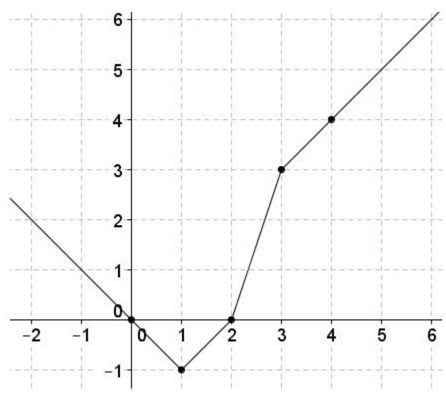
$$A^2 - B^2 + AB - B^2 = (A - B)(A + B) + B(A - B) = (A - B)(A + 2B) = 0.$$

Therefore, $A - B = 0$ or $A + 2B = 0$. Combining this with what we got from the first equation, we have $A = X/2 - 4/5$ or $A = 2X - 16/5$. The sum of the possible values of $A$ is $5X/2 - 4$. After finding that $X = 2$ from problem 29, we have the sum of all possible $A$ equal to $5(2)/2 - 4 = 1$.

29. Answer: $\boxed{2}$ Therefore, W=5X/2-4. If we call $AB = m$, we either have $CD = 2m$ or $CD = m/2$. If it were the former case, then $m + 2W > 2m$, so $2W > m$. If it were the latter case, then $m/2 + 2W > m$, so $4W > m$. Since the trapezoid is unique, $m$ must satisfy only one inequality. Since $4W$ is always greater than $2W$ when they are both positive (side lengths must be positive), we have $4W > m \geq 2W$.

From the previous question, we have $W = 5X/2 - 4$. Since $m$ is the answer to this question, we know that $m = X$, since $X$ is the answer to question 29 in problem 28. Therefore, $X \geq 5X - 8$ so $2 \geq X$. The maximum value of $X$ is 2. Plugging 2 into $10X - 6 > m = X$, we see this holds true. Thus the answer is 2.

30. Answer: $\boxed{1}$ Plugging in the values from above, our expression becomes $|Z-1|+|Z-2|-|3-Z|$. We consider the 4 regions when $Z \leq 1$, $1 \leq Z \leq 2$, $2 \leq Z \leq 3$, and $3 \leq Z$. As $Z$ approaches 1 from below, the expression decreases so $Z = 1$ gives a minimum. As $Z$ approaches 2 from 1, $|Z-1|+|Z-2|$ is constant and $|3-Z|$ decreases so the expression increases. As $Z$ approaches 3 from 2, the expression increases. As $Z$ grows above 3, the expression continues to increase. Thus, the expression is at a minimum for $Z = 1$.




31. Answer: $\boxed{3363}$ Since the horizon goes back to the original elevation, there must be an equal number of segments with slope 1 and segments with slope $-1$. If there are $n$ pairs of these segments, there are $2^n$ ways to create them: each pair of segments can either go up then down or down then up. We then need to choose $2n$ segments to place them in, so for each $n$, there are $2^n\binom{10}{2n}$ ways to create the horizon, where $n$ can be 0, 1, 2, 3, 4, or 5. The answer is

$$2^0\binom{10}{0} + 2^1\binom{10}{2} + 2^2\binom{10}{4} + 2^3\binom{10}{6} + 2^4\binom{10}{8} + 2^5\binom{10}{10} = 3363.$$

32. Answer: $\boxed{936}$ There are three consecutive strings of 4 base pairs in a DNA sequence of 6 pairs (either base pairs 1-2-3-4, 2-3-4-5, or 3-4-5-6). We consider the three cases in which either 1, 2, or 3 of these strings has an A, T, C, G in some order.

Case 1: All 3 of the strings contain each of A, T, C, G: the rst 4 base pairs uniquely determine the whole sequence. There are $4! = 24$ such sequences of 4 base pairs.

Case 2: 2 of the three strings contain each of A, T, C, G: then either it is sequences 1-2-3-4 and 3-4-5-6 that contain the A, T, C, G, or sequences 1-2-3-4 and 2-3-4-5, or sequences 2-3-4-5 and 3-4-5-6. For the first possibility, the first 4 base pairs uniquely determine the sequence, giving us $4! = 24$ sequences. For the two latter possibilities, there are $4! \cdot 3 = 72$ DNA sequences that work. Thus, the total number of DNA sequences included in this second possibility is $72 + 72 + 24 = 168$.

Case 3: 1 of the three strings contain each of an A, T, C, G: if this string is 1-2-3-4 or 3-4-5-6, there are $4! \cdot 11 = 264$ DNA sequences. If this string is 2-3-4-5, there are $4! \cdot 3 \cdot 3 = 216$

DNA sequences. Thus, there are $264 + 264 + 216 = 744$ DNA sequences included in this third possibility.

Overall, there are $24 + 168 + 744 = 936$ stunningly nondescript sequences of 6 base pairs.

33. Answer: $\boxed{22 - 8\sqrt{7}}$ Let $x = 3 + 2s + 3t$, $y = 7 - 2t$, and $z = 5 - 2s - t$. Then $x^2 + y^2 + z^2 = 83$ and $x + y + z = 15$, so $(x, y, z)$ lies on the intersection of a sphere centered at the origin and a plane symmetric in $x$,$y$,$z$, which is a circle. The center of the circle is thus (5,5,5). Let $A$ be the center of the sphere, $B$ be the center of the circle and $C$ be a point on the circle. Then, $\overline{AB}$ is perpendicular to the plane of the circle so $ABC$ is a right triangle. The length of $\overline{AB}$ is the space diagonal of a $5 \times 5 \times 5$ cube which is $5\sqrt{3}$. Therefore, the radius of the circle is $BC = \sqrt{83 - (5\sqrt{3})^2} = 2\sqrt{2}$.

    The expression we wish to minimize can be written as $(x - 8)^2 + (y - 4)^2 + (z - 3)^2$, which is the squared distance between $(x, y, z)$ and $(8, 4, 3)$, which also lies on the plane $x + y + z = 15$. Thus, the shortest possible distance between the two points is the distance from $(8, 4, 3)$ to the center of the circle minus the radius: $\sqrt{(8 - 5)^2 + (4 - 5)^2 + (3 - 5)^2} - 2\sqrt{2} = \sqrt{14} - 2\sqrt{2}$. The squared distance is then $22 - 8\sqrt{7}$.

34. Answer: $\boxed{1360}$ While this was part of the guessing/estimation round, this can be done exactly using logarithms. Let $a$ be the number of $n$ such that $f(n) = 3$ and and $b$ be the number of $n$ such that $f(n) = 4$, with $n \leq 2013$. It is clear that for each $n$, $f(n) = 3$ or $f(n) = 4$, so we have $a + b = 2013$. Counting the number of powers of 2 with at most 2013 digits, we have $2^x < 10^{2013}$, so $x \log_{10} 2 \leq 2013$, or $x \leq \left\lfloor \frac{2013}{\log_{10} 2} \right\rfloor = 6687$. Therefore, x can range from 0 to 6687 inclusive so $3a + 4b = 6687$ since $3a + 4b$, by definition, counts the number of powers of 2 with at most 2013 digits. Solving this for $a$, we have $a = 1360$.

35. Answer: $\boxed{30,178}$ The total number of characters in the source files is 30,178.

36. Answer: $\boxed{?}$ Answers will depend on competitors' responses.