4th Annual Lexington Mathematical Tournament - Guts Round Part 1
March 30th, 2013

1. [**5**] How many powers of 2 are greater than 3 but less than 2013?

2. [**5**] What number is equal to six greater than three times the answer to this question?

3. [**5**] Surya Cup-a-tea-raju goes to Starbucks Coffee to sip coffee out of a styrofoam cup. The cup is a cylinder, open on one end, with base radius 3 centimeters and height 10 centimeters. What is the exterior surface area of the styrofoam cup?

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4. [**5**] Andrew has two 6-foot-length sticks that he wishes to make into two of the sides of the entrance to his fort, with the ground being the third side. If he wants to make his entrance in the shape of a triangle, what is the largest area that he can make the entrance?

5. [**5**] Ethan and Devin met a fairy who told them "if you have less than 15 dollars, I will give you cake". If both had integral amounts of dollars, and Devin had 5 more dollars than Ethan, but only Ethan got cake, how many different amounts of money could Ethan have had?

6. [**5**] If $2012^x = 2013$, for what value of a, in terms of x, is it true that $2012^a = 2013^2$?

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7. [**6**] Find the ordered triple (L, M, T) of positive integers that makes the following equation true:
$$1 + \frac{1}{L + \frac{1}{M + \frac{1}{T}}} = \frac{79}{43}.$$

8. [**6**] Jonathan would like to start a banana plantation so he is saving up to buy an acre of land, which costs \$600,000. He deposits \$300,000 in the bank, which gives 20% interest compounded at the end of each year. At this rate, how many years will Jonathan have to wait until he can buy the acre of land?

9. [**6**] Arul and Ethan went swimming at their town pool and started to swim laps to see who was in better shape. After one hour of swimming at their own paces, Ethan completed 32 more laps than Arul. However, after that, Ethan got tired and swam at half his original speed while Arul's speed didn't change. After one more hour, Arul swam a total of 320 laps. How many laps did Ethan swim after two hours?

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10. [**6**] A right triangle with a side length of 6 and a hypotenuse of 10 has circles of radius 1 centered at each vertex. What is the area of the space inside the triangle but outside all three circles?

11. [**6**] In isosceles trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$ and the lengths of $\overline{AB}$ and $\overline{CD}$ are 2 and 6, respectively. Let the diagonals of the trapezoid intersect at point E. If the distance from E to $\overline{CD}$ is 9, what is the area of triangle ABE?

12. [**6**] If 144 unit cubes are glued together to form a rectangular prism and the perimeter of the base is 54 units, what is the height?

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13. [**7**] Given that $x^3 + y^3 = 208$ and $x + y = 4$, what is the value of $\frac{1}{x} + \frac{1}{y}$?

14. [**7**] Find the sum of all three-digit integers n such that the value of n is equal to the sum of the factorials of n's digits.

15. [**7**] Three christmas lights are initially off. The Grinch decides to fiddle around with the lights, switching one of the lights each second. He wishes to get every possible combination of lights. After how many seconds can the Grinch complete his task?

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16. [**7**] A regular tetrahedron of side length 1 has four similar tetrahedrons of side length 1/2 chopped off, one from each of the four vertices. What is the sum of the numbers of vertices, edges, and faces of the remaining solid?

17. [**7**] Mario serves a pie in the shape of a regular 2013-gon. To make each slice, he must cut in a straight line starting from one vertex and ending at another vertex of the pie. Every vertex of a slice must be a vertex of the original 2013-gon. If every person eats at least one slice of pie regardless of the size, what is the maximum number of people the 2013-gon pie can feed?

18. [**7**] Find the largest integer x such that $x^2 + 1$ divides $x^3 + x - 1000$.

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19. [**8**] In $\triangle ABC$, $\angle B = 87°$, $\angle C = 29°$, and $AC = 37$. The perpendicular bisector of $\overline{BC}$ meets $\overline{AC}$ at point T. What is the value of $AB + BT$?

20. [**8**] Consider the sequence $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1+3}{2}$, $f(4) = \frac{1+3}{2+4}$, $f(5) = \frac{1+3+5}{2+4} \ldots$ What is the minimum value of n, with $n > 1$, such that $|f(n) - 1| \leq \frac{1}{10}$.

21. [**8**] Three unit circles are centered at $(0, 0), (0, 2)$, and $(2, 0)$. A line is drawn passing through $(0, 1)$ such that the region inside the circles and above the line has the same area as the region inside the circles and below the line. What is the equation of this line in $y = mx + b$ form?

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22. [**8**] The two walls of a pinball machine are positioned at a 45 degree angle to each other. A pinball, represented by a point, is fired at a wall (but not at the intersection of the two walls). What is the maximum number of times the ball can bounce off the walls?

23. [**8**] Albert is fooling people with his weighted coin at a carnival. He asks his guests to guess how many times heads will show up if he flips the coin 4 times. Richard decides to play the game and guesses that heads will show up 2 times. In the previous game, Zach guessed that the heads would show up 3 times. In Zach's game, there were least 3 heads, and given this information, Zach had a $\frac{4}{9}$ chance of winning. What is the probability that Richard guesses correctly?

24. [**8**] Let S be the set of all positive integers relatively prime to 2013 that have no prime factor greater than 15. Find the sum of the reciprocals of all of the elements of S.

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25. [**9**] Define a *hilly* number to be a number with distinct digits such that when its digits are read from left to right, they strictly increase, then strictly decrease. For example, 483 and 1230 are both *hilly* numbers, but 123 and 1212 are not. How many 5-digit *hilly* numbers are there?

26. [**9**] Triangle ABC has $AB = 4$ and $AC = 6$. Let the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$ be D and the foot of the perpendicular from C to the angle bisector of $\angle BAC$ be E. What is the value of AD/AE?

27. [**9**] Given that $(7+4\sqrt{3})^x + (7-4\sqrt{3})^x = 10$, find all possible values of $(7+4\sqrt{3})^x - (7-4\sqrt{3})^x$.

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Note: In this set, the answers for each problem rely on answers to the other problems.

28. [**11**] Let X be the answer to question 29. If $5A + 5B = 5X - 8$ and $A^2 + AB - 2B^2 = 0$, find the sum of all possible values of A.

29. [**11**] Let W be the answer to question 28. In isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, line segments $\overline{AC}$ and $\overline{BD}$ split each other in the ratio 2 : 1. Given that the length of $\overline{BC}$ is W, what is the greatest possible length of $\overline{AB}$ for which there is only one trapezoid $ABCD$ satisfying the given conditions?

30. [**11**] Let W be the answer to question 28 and X be the answer to question 29. For what value of Z is $|Z - X| + |Z - W| - |W + X - Z|$ at a minimum?

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31. [**13**] Peijin wants to draw the horizon of Yellowstone Park, but he forgot what it looked like. He remembers that the horizon was a string of 10 segments, each one either increasing with slope 1, remaining flat, or decreasing with slope 1. Given that the horizon never dipped more than 1 unit below or rose more than 1 unit above the starting point and that it returned to the starting elevation, how many possible pictures can Peijin draw?

32. [**13**] DNA sequences are long strings of A, T, C, and G, called base pairs. (e.g. AATGCA is a DNA sequence of 6 base pairs). A DNA sequence is called *stunningly nondescript* if it contains each of A, T, C, G, in some order, in 4 consecutive base pairs somewhere in the sequence. Find the number of *stunningly nondescript* DNA sequences of 6 base pairs (the example above is to be included in this count).

33. [**13**] Given variables s, t that satisfy $(3 + 2s + 3t)^2 + (7 - 2t)^2 + (5 - 2s - t)^2 = 83$, find the minimum possible value of $(-5 + 2s + 3t)^2 + (3 - 2t)^2 + (2 - 2s - t)^2$.

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34. [$\leq$ **15**] Let $f(n)$ be the number of powers of 2 with n digits. For how many values of n from 1 to 2013 inclusive does $f(n) = 3$? If your answer is N and the actual answer is C, then the score you will receive on this problem is $\max\{15 - \frac{|N-C|^2}{6039}, 0\}$, rounded to the nearest integer.

35. [$\leq$ **15**] How many total characters are there in the source files for the LMT 2013 problems? If your answer is N and the actual answer is C, then the score you receive on this problem is $\max\{15 - \dfrac{|N - C|}{1337}, 0\}$, rounded to the nearest integer.

36. [$\leq$ **15**] Write down two distinct integers between 0 and 300, inclusive. Let S be the collection of everyone's guesses. Let x be the smallest nonnegative difference between one of your guesses and another guess in S (possibly your other guess). Your team will be awarded $\min(15, x)$ points.