3rd Annual Lexington Mathematical Tournament
Team Round

Solutions

1 Potpourri

1. Answer: $\boxed{77}$

   Solution:

2. Answer: $\boxed{3}$

   Solution: Let a, b, and c be the radii of circles A, B, and C, respectively. Then, $a+b=8$, $a+c=10$, and $b+c=12$. Solving gives $a=3$, $b=5$, and $c=7$.

3. Answer: $\boxed{31}$

   Solution: Solving $\frac{1}{x}+\frac{2}{2y+1}=\frac{1}{y}$ for x, we get $\frac{1}{x}=\frac{1}{y}-\frac{2}{2y+1}=\frac{2y+1}{y(2y+1)}-\frac{2y}{y(2y+1)}=\frac{1}{2y^2+1} \Rightarrow x=2y^2+1$. Since $x<2012$, $2y^2+1<2012 \Rightarrow y^2<1005.5$. There are 31 square numbers y^2 that satisfy this inequality, so there are 31 possible valies of x.

4. Answer: $\boxed{-1 \text{ and } 19}$

   Solution: We have that $|8-2x|=11+x$. If $8-2x \geq 0 \Rightarrow x \leq 4$, then $8-2x=11+x \Rightarrow x=-1$. If $8-2x<0 \Rightarrow x>4$, then $-(8-2x)=11+x \Rightarrow x=19$. Therefore, the two solutions are $x=-1$ and $x=19$.

5. Answer: $\boxed{-\dfrac{4}{9}}$

   Solution: The line with slope 4 intersects the x-axis at $(2,0)$. Let the other line intersect the y-axis at $(0,k)$. Then, by dividing the left-most region by cutting it at the line $y=4$, we see that the area of the region is the sum of the area of a trapezoid, which is 10, and the area of a triangle, which is $\frac{3}{2}(k-4)$. The area of this region must equal $\frac{1}{4}$ of that of the entire region, or $\frac{1}{4}\cdot 48=12$. Therefore, we have $10+\frac{3}{2}(k-4)=12 \Rightarrow k=\frac{16}{3}$. Finally, from the points $(0,\frac{16}{3})$ and $(3,4)$, we find that the slope of the line is $\frac{4-16/3}{3-0}=-\frac{4}{9}$.

6. Answer: $\boxed{10}$

   Solution: The sum of the angles of a polygon with n sides is $(n-2)\cdot 180$. Since the sum of the angles of the polygon is composed of nonzero multiples of 138 and 150, we have $138n<(n-2)\cdot 180<150n$. Solving this inequality, we get $\frac{60}{7}<n<12$. Let there be x angles of degree 138. Then, we have $138x+150(n-x)\equiv 0 \pmod{180} \Rightarrow 23x+25(n-x)=25n-2x\equiv 0 \pmod{30}$. It follows that n must be even. Thus, the only possible value for n is 10. (The polygon has five of each angle.)

7. Answer: $\boxed{514.5}$

   Solution: Since $LMT \geq 100$, $MATH \leq 1912$, which means that M must be 1. Then we express the sum in terms of the digits: $1000M+100A+10T+H+100L+10M+T=1010+100A+100L+10T+H=2012 \Rightarrow 100(A+L)+11T+H=1002$. Taking this mod 100 gives $11T+H\equiv 2 \pmod{100}$. Since $0\leq 11T+H\leq 11(99)+(9)=108$, $11T+H$ can either be 2 or 102. If $11T+H=2$, then $T=0$ and $A+L=10$, so the possible values of LMT are $110,210,310,\ldots,910$. If $11T+H=102$, when $T=9$ and $A+L=9$, so the possible values of LMT are $119,219,319,\ldots,919$. These values have an average of 514.5.

8. Answer: $\boxed{2}$

   Solution: Let O be the center and X and Y be two points on side S of the larger square. For X and Y to be vertices of the two smaller squares, we must have OX and OY be half the length of the diagonals of the smaller squares, or $\frac{1}{2} \cdot \sqrt{7} \cdot \sqrt{2} = \frac{\sqrt{14}}{2}$. Let the perpendicular from O to side S intersect S at P. Then, $OP = \frac{\sqrt{10}}{2}$. By the Pythagorean theorem, $OP^2 + PX^2 = OX^2 \Rightarrow PX = \sqrt{OX^2 - OP^2} = \sqrt{(\frac{\sqrt{14}}{2})^2 - (\frac{\sqrt{10}}{2})^2} = 1$. Similarly, $OP^2 + PY^2 = OY^2 \Rightarrow PY = 1$. Thus, the distance between X and Y is $PX + PY = 2$.

9. Answer: $\boxed{120127}$

   Solution: There is only one four-digit number with a substring of 2012. For five-digit numbers, we can either place the remaining digit ahead of or behind 2012. We have 9 choices for placing a digit ahead and 10 for behind. This makes 20 numbers so far. Then, we list the ten smallest six-digit numbers: 102012, 112012, 120120, 120121, 120122, 120123, 120124, 120125, 120126, 120127. Therefore, the 30th number in the sequence is 120127.

10. Answer: $\boxed{2101}$

    Solution: Let a, b, and c be the number of times the terms x, $\sqrt{x}$, and $\sqrt[3]{x}$ are used, respectively. Then, we look for the solutions of $a + b + c = 10$ and $a + \frac{b}{2} + \frac{c}{3} = 5$. Substituting $10 - b - c$ for a in the second equation gives $10 - b - c + \frac{b}{2} + \frac{c}{3} = 5 \Rightarrow 3b + 4c = 30$. Taking this equation mod 3 gives $c \equiv 0 \pmod{3}$. If $c = 0$, then $b = 10$ and $a = 0$. If $c = 3$, then $b = 6$ and $a = 1$. If $c = 6$, then $b = 2$ and $a = 2$. The number of ways to choose a x terms, b $\sqrt{x}$ terms, and c $\sqrt[3]{x}$ terms from ten expressions of $x + \sqrt{x} + \sqrt[3]{x}$ is $\binom{10}{10-a}\binom{10-a}{b}$. Therefore, there are $\binom{10}{0}\binom{10}{10} + \binom{10}{1}\binom{9}{6} + \binom{10}{2}\binom{8}{2} = 2101$ possible ways.

2 Long Answer

2.1 The 7 Divisibility Rule

1. Let $d \mid N$ denote that N is divisible by d. Let $N = 10m + n \equiv r \pmod 7$ where n is a single-digit integer. After applying the divisibility rule, we get $m - 2n \equiv 10(m - 2n) = 10m - 20n = N - 21n \equiv N \pmod 7$ because 10 and 7 are relatively prime and because $7 \mid 21n$. Since the two integers always have the same residue mod 7, $7 \mid N$ if and only if $7 \mid N - 21n$.

2. Let $N = 10m + n \equiv r \pmod d$ where n is a single-digit integer. After applying the rule, we get $m - kn \equiv 10(m - kn) = 10m - 10kn = N - (10k + 1)n \pmod d$ because 10 and d are relatively prime. Since $d \mid N \Rightarrow d \mid N - (10k + 1)n$, we must have $d \mid (10k + 1)n$. Since n can be any digit, letting $n = 1$ gives $d \mid (10k + 1)$. Therefore, k must satisfy $d \mid (10k + 1)$. Now we show that such a k works. Let $N = 10m + n \equiv r \pmod d$ where n is a single-digit integer. After applying the divisibility rule, we get $m - kn \equiv 10(m - kn) = N - (10k + 1)n \equiv N$ (mod 7) because $d \mid (10k + 1)n$. Since the two integers always have the same residue mod d, $d \mid N$ if and only if $d \mid N - (10k + 1)n$.

   (a) We find that $k = 1$ is the smallest value of k for which $11 \mid 10k + 1$. Therefore, $k = \boxed{1}$.

   (b) We find that $k = 5$ is the smallest value of k for which $17 \mid 10k + 1$. Therefore, $k = \boxed{5}$.

2.2 Solving Cubic Equations

3. Substituting $a - b$ for x and $a^3 - b^3$ for the 12 on the right-hand side gives $(a - b)^3 + 12(a - b) = a^3 - b^3 \Rightarrow a^3 - 3a^2b + 3ab^2 - b^3 + 12(a - b) = a^3 - b^3 \Rightarrow 12(a - b) = 3ab(a - b) \Rightarrow ab = \boxed{4}$ (since $a - b \neq 0$).

4. We have that $ab = 4 \Rightarrow a^3b^3 = 64$ and that $a^3 - b^3 = 12$. Substituting $\frac{64}{a^3}$ for b^3 in the second equation gives $a^3 - \frac{64}{a^3} = 12 \Rightarrow (a^3)^2 - 12a^3 - 64 = 0$. This equation is a quadratic in terms of a^3 and factors to $(a^3 + 4)(a^3 - 16) = 0 \Rightarrow a = -\sqrt[3]{3}$ or $\sqrt[3]{16}$. Since we need to only find one solution, let $a = \sqrt[3]{16}$. (It turns out that if we let $a = -\sqrt[3]{3}$, we will get the same value of x, just in a slightly different form. In fact, the cubic equation only has one real solution.) Then, $b^3 = a^3 - 12 = 4 \Rightarrow b = \sqrt[3]{4}$. Thus, $x = \sqrt[3]{16} - \sqrt[3]{4} = \boxed{2\sqrt[3]{2} - \sqrt[3]{4}}$.

5. Again, we substitute $a - b$ for x and $a^3 - b^3$ for the q on the right-hand side. Then, $(a - b)^3 + p(a - b) = a^3 - b^3 \Rightarrow p(a - b) = 3ab(a - b) \Rightarrow ab = \frac{p}{3} \Rightarrow a^3b^3 = \frac{p^3}{27}$. Substituting $\frac{p^3}{27a^3}$ for b^3 in $a^3 - b^3 = q$ gives $a^3 - \frac{p^3}{27a^3} = q \Rightarrow (a^3)^2 - qa^3 - \frac{p^3}{27} = 0 \Rightarrow a^3 = \frac{q \pm \sqrt{q^2 - \frac{4p^3}{27}}}{2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. Let $a^3 = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. Then, $b^3 = a^3 - q \Rightarrow b^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. Thus, $x = a - b = \boxed{\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$. Note that there are many different ways to rewrite this expression. We will give credit to all equivalent expressions.

2.3 Tiles and Dominos

6. (a) We can either use no dominos or one domino. There is 1 way to use no dominos, and 2 ways to use one domino because we can put the remaining tile to the left of or to the right of the domino. Therefore, there are $\boxed{3}$ ways in total.

   (b) We can either use 0, 1, or 2 dominos. There is 1 way to use 0 dominos. If we use 1 domino, then we have 3 ways to position the domino and fill the rest of the empty squares with tiles. If we use 2 dominos, there is only one way because the two dominos fill up the entire rectangle. In total, there are $\boxed{5}$ ways.

7. If the last square is occupied by a tile, then we can tile the remaining $n-1$ squares in $f(n-1)$ ways. If the last square is occupied by a domino, then we can tile the remaining $n-2$ squares in $f(n-2)$ ways. Furthermore, we do not overcount or miss any possibilities because every fitting ends in either a tile or a domino. Therefore, $f(n) = f(n-1) + f(n-2)$.

8. Using our recursive definition in the last problem, we get $f(5) = 8$, $f(6) = 13$, $f(7) = 21$, $f(8) = 34$, $f(9) = 55$, and $f(10) = \boxed{89}$.

9. $f(1) = 1$ because, clearly, there is only one way to fill in a 1×1 rectangle. $f(2) = 2$ because $f(3) = f(1) + f(2) \Rightarrow f(2) = f(3) - f(1) = 3 - 1 = 2$. Therefore, we have that $f(1) = c((a+b)^2 - (a-b)^2) = 4abc = 1$, $f(2) = c((a+b)^3 - (a-b)^3) = bc(6a^2 + 2b^2) = 2$, and $f(3) = c((a+b)^4 - (a-b)^4) = 8abc(a^2+b^2) = 3$. From here, we know that $abc = \frac{1}{4}$, $a \cdot f(2) = abc(6a^2 + b^2) = 2a \Rightarrow \frac{1}{4}(6a^2 + 2b^2) = 2a \Rightarrow b^2 = -3a^2 + 4a$, and $f(3) = 8abc(a^2+b^2) = 2(a^2+b^2) = 3 \Rightarrow b^2 = \frac{3}{2} - a^2$. Setting $-3a^2 + 4a = b^2 = \frac{3}{2} - a^2$ gives $4a^2 - 8a + 3 = (2a-1)(2a-3) = 0 \Rightarrow a = \frac{1}{2}$ or $\frac{3}{2}$. However, $a^2 + b^2 = \frac{3}{2} \Rightarrow a^2 \leq \frac{3}{2}$, so a must be $\frac{1}{2}$. Then, $b^2 = \frac{3}{2} - a^2 = \frac{5}{4} \Rightarrow b = \frac{\sqrt{5}}{2}$ and $c = \frac{1}{4ab} = \frac{1}{\sqrt{5}}$. Thus, $\boxed{a = \frac{1}{2}}$, $\boxed{b = \frac{\sqrt{5}}{2}}$, and $\boxed{c = \frac{1}{\sqrt{5}}}$.