3rd Annual Lexington Mathematical Tournament
Guts Round

Solutions

1. Answer: $\boxed{99}$

   Solution: If x is increased by 10%, then the new value is $1.1x$. If x is decreased by 10%, then the new value is $0.9x$. Therefore, the current price of the TV is $100 \cdot 1.1 \cdot 0.9 = 99$.

2. Answer: $\boxed{503}$

   Solution: We see that $(9w+8x+7y)+(w+2x+3y) = 10w+10x+10y = 42+8 = 50 \Rightarrow w+x+y = 5$. Then we can write $100w + 101x + 102y = 99(w + x + y) + w + 2x + 3y = 99(5) + 8 = 503$.

3. Answer: $\boxed{16}$

   Solution: Since both 37 and 41 are prime, the factor must be the product of an element from $\{37^0, 37^1, 37^2, 37^3\}$ and an element from $\{41^0, 41^1, 41^2, 41^3\}$. There are $4 \cdot 4 = 16$ such factors.

4. Answer: $\boxed{2}$

   Solution: Every hour, the three hoses together fill up $\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$ of the pool. Thus, the pool is filled in 2 hours.

5. Answer: $\boxed{\pi}$

   Solution: We draw the line that perpendicularly bisects the diameters of both the largest and second-largest semicircles. Also, we draw a radius of the largest semicircle that coincides with one of the two intersection points between the diameter of the second-largest semicircle and the arc of the largest. From the isosceles right triangle on the resulting diagram, we find the ratio of the radii of the second-largest semicircle to that of the largest to be $\frac{1}{\sqrt{2}}$. So the ratio of the areas is $\frac{1}{2}$. From here, we get that the ratio of the area of the third-largest semicircle to that of the second-largest is $\frac{1}{2}$, and so on. The area of the largest semicircle is $\frac{1}{2}\pi$. Therefore, the sum of all the areas is $\frac{1}{2}\pi + \frac{1}{2} \cdot \frac{1}{2}\pi + \frac{1}{4} \cdot \frac{1}{2}\pi + \cdots = \frac{1}{2}\pi(\frac{1}{2} + \frac{1}{4} + \cdots) = \frac{1}{2}\pi(\frac{1}{1-\frac{1}{2}}) = \pi$.

6. Answer: $\boxed{6036}$

   Solution: Since 1 and 2 are roots of $P(x)$, the polynomial must be of the form $a(x-1)(x-2)$. Setting $P(0) = 2012$ gives $2a = 2012 \Rightarrow a = 1006$. Thus, $P(-1) = 1006((-1)-1)((-1)-2) = 6036$.

7. Answer: $\boxed{20}$

   Solution: We need to find the locus of the midpoints of $\overline{AB}$, where B is another point on the circumference of the circle. After some experimentation, we observe that the midpoints form a smaller circle with half the radius. Indeed, if the circle assumes the equation $x^2 + (y-r)^2 = r^2$ and we let A be at the origin, we can divide this equation by 4 to get $(\frac{x}{2})^2 + (\frac{y}{2} - \frac{r}{2})^2 = (\frac{r}{2})^2$. This means that for each $B = (x, y)$ on the circle, the midpoint of $\overline{AB}$, $(\frac{x}{2}, \frac{y}{2})$, lies on a circle with radius $\frac{r}{2}$. Therefore, the smaller circle has area 20, which is $\frac{1}{4}$ of the area of the original circle.

8. Answer: $\boxed{12\pi}$

   Solution: Assume that the rectangle starts on a side of length 6, that it is rolling clockwise to the right, and that we follow the bottom-left corner at the beginning. Since the rectangle's perimeter is 28 feet, the rectangle makes one complete rotation in 28 feet. In the first, second, and third turn, the point that we are following moves in a quarter-circle if radius 6, 10, and 8, respectively. The point does not move during the last turn. Therefore, the point moved $\frac{1}{4} \cdot 2\pi(6) + \frac{1}{4} \cdot 2\pi(10) + \frac{1}{4} \cdot 2\pi(8) = 12\pi$.

9. Answer: $\boxed{44}$

Solution: In one hour, the hour hand moves $30°$ and the minute hand moves $360°$. So, in h hours, the hour hand and minute hand move $30h$ and $360h$ degrees, respectively. For the two hands to end up 90 degrees apart, we must have either $30h - 360h \equiv 90 \pmod{360}$ or $360h - 30h \equiv 90 \pmod{360}$. Therefore, $330h = 360k \pm 90$ for integer k. Since $0 \le h \le 24$, we must have $0 \le 360k \pm 90 \le 24 \cdot 330$. For each of the two cases, there are 22 possible values of k, giving a total of 44 possible values of h. Therefore, the two hands are orthogonal 44 times.

10-12. Answer: $\boxed{5, 17, 89}$

Solution: Let the length of half of AD be s. Then, the side length of the square formed by the midpoints is $s\sqrt{2}$. We have that the ratio of the area of $ADGH$ to the area of the square formed by the midpoints is $\frac{(2s)^2}{(s\sqrt{2}}^2 = 2$, so $ADGH$ has area $2B$. By the Pythagorean theorem, $BC = \sqrt{AB^2 - AC^2} = \sqrt{2B - B} = \sqrt{B}$. Therefore, $A = \lfloor \sqrt{B} \rfloor$. From here, we try $A = 1, 2, \ldots$ for problem 12 until everything works out, which is when $A = 5$, $B = 17$, and $C = 89$.

13. Answer: $\boxed{438}$

Solution: $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[6]{6} = \sqrt[n]{m}$ can be expressed as $2^{\frac{2}{3}} \times 3^{\frac{1}{2}} = m^{\frac{1}{n}}$. Raising both sides to the nth power gives $2^{\frac{2}{3}n} \times 3^{\frac{1}{2}n} = m$. For m to be an integer, both $\frac{2}{3}n$ and $\frac{1}{2}n$ must be integers. The smallest possible n that makes this true is $\operatorname{lcm}(3, 2) = 6$. Then, $m = 2^{\frac{2}{3}(6)} \times 3^{\frac{1}{2}(6)} = 432 \Rightarrow m + n = 438$.

14. Answer: $\boxed{\dfrac{23}{364}}$

Solution: Assume that every candy is distinguishable. There are $\binom{5}{3} \cdot 11 = 110$ ways to choose three garlic-flavored candies and one of another flavor, and $\binom{5}{4} = 5$ ways to choose four garlic-flavored candies. There are $\binom{16}{4} = 1820$ total ways to choose four candies, so the answer is $\frac{110+5}{1820} = \frac{23}{364}$.

15. Answer: $\boxed{\dfrac{29}{36}\pi}$

Solution: Since the quadrilateral is symmetric across $\overline{BD}$, so is the circumcircle, so $\overline{BD}$ is a diameter of that circle. Also, this means that $\angle BCD$ and $\angle BAD$ are right angles. Then, $BD = \sqrt{BC^2 + CD^2} = \sqrt{5}$ by the Pythagorean theorem, and the radius of the circumcircle is $\frac{\sqrt{5}}{2}$. The incircle of $ABCD$ is also symmetric across $\overline{BD}$, so its center lies on $\overline{BD}$. Drawing the radii to where the circle is tangent to sides $\overline{BC}$ and $\overline{CD}$, we have similar triangles resulting in the proportion $\frac{r}{AB - r} = \frac{AD}{AB}$, where r is the radius of the incircle. Solving, we get $r = \frac{2}{3}$. Thus, the area between the two circles is $\pi(\frac{\sqrt{5}}{2})^2 - \pi(\frac{2}{3})^2 = \frac{29}{36}\pi$.

16. Answer: $\boxed{0 \text{ and } \dfrac{-1 \pm \sqrt{5}}{2}}$

Solution: We have $x = 0$ as one solution. Then, assuming that $x \neq 0$, we divide both sides of the equation by $\frac{[}{3}]x$ to get $x^2 + x = 1$. Applying the quadratic formula, we get that $x = \frac{-1 \pm \sqrt{5}}{2}$. Thus, the three solutions are 0 and $\frac{-1 \pm \sqrt{5}}{2}$.

17. Answer: $\boxed{46 - 32\sqrt{2}}$

Solution: If we want to remove the least possible area, we will end up with the regular octagon with the biggest possible area. We know that this octagon must fit within the square bounded by $x = \pm 2$ and $y = \pm 2$. The largest possible regular octagon therefore has four sides that coincide with the sides of the square. Let the side length of the octagon be s. The sides of the octagon that do not coincide with sides of the square are the hypotenuses of isosceles right triangles, each with legs of length $\frac{s}{\sqrt{2}}$. Therefore, the side length of the square is $\frac{s}{\sqrt{2}}+s+\frac{s}{\sqrt{2}}=s(1+\sqrt{2})$. Setting this expression equal to 4, we get $s=\frac{4}{1+\sqrt{2}}$. To determine the area of this regular octagon, we subtract the four isosceles triangles from the square to obtain $4^2-4\cdot\frac{1}{2}\cdot\frac{s}{\sqrt{2}}=16-s^2=16-\frac{16}{3+2\sqrt{2}}=16-\frac{16(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})}=16-\frac{16(3-2\sqrt{2})}{3^2-(2\sqrt{2})^2}=32\sqrt{2}-32$. The difference between this area and the original octagon, which has area 14, is $14-(32\sqrt{2}-32)=46-32\sqrt{2}$.

18. Answer: $\boxed{2252}$

    Solution: Let the number be expressed as $ABCD$, where each letter represents a digit. Then, we know that $(10A+B)+(10B+C)+(10C+D)=99 \Rightarrow 10A+11B+11C+D=99$. Taking this equation mod 11 gives $10A+D\equiv 0 \Rightarrow -A+D\equiv 0 \Rightarrow A\equiv D \pmod{11}$. However, since it is impossible for A and D to differ by 11 or more, we must have $A=D$. Since $ABCD$ is divisible by 8, D must be even, and since $A\neq 0$, we try $A=D=2$. Then, $B+C=7$. Testing out values for B and C, we find that 2252 is the lowest possible valid number.

19. Answer: $\boxed{10}$

    Solution: Let the sum of the number of kids who enjoy one subject exclusively and the number of kids who enjoy all three subjects be x. Also, let the number of kids who only enjoy geometry and number theory, the number of kids who only enjoy geometry and algebra, and the number of kids who only enjoy number theory and algebra be a, b, and c, respectively. Then, we have the equations $x+a+b=30$, $x+a+c=40$, and $x+b+c=50$. Solving this system of equations for a, b, and c gives $(a,b,c)=(10-\frac{x}{2},20-\frac{x}{2},30-\frac{x}{2})$. Therefore, the maximum value of b, the number we desire, is 20, when $x=0$. Furthermore, since $b=a+10$, the minimum possible b is 10. It can be verified that both of these values are valid. Thus, the difference between minimum and maximum is $20-10=10$.

20. Answer: $\boxed{125}$

    Solution: The number of ways to move from (x,y) to $(x+h,y+k)$ by traveling the shortest path possible along the lattice grid is $\binom{h+k}{k}$, because out of the $(h+k)$ moves of 1 unit each, we must choose h of them to be rightward. If the rock is placed at (x,y), then the number of paths blocked is the number of ways to travel from $(0,0)$ to (x,y) times the number of ways to travel from (x,y) to $(6,4)$, which is $\binom{x+y}{x}\binom{(4-x)+(6-y)}{4-x}=\binom{x+y}{x}\binom{(10-x-y)}{4-x}$. Through trial and error, the maximum value of the above expression is 126 when $(x,y)=(5,4)$ or $(1,0)$, and the minimum value is 1 when $(x,y)=(0,6)$ or $(4,0)$. Therefore, the difference between the maximum and minimum number of blocked paths, which is also the difference between the maximum and minimum number of available paths, is $126-1=125$.

21. Answer: $\boxed{42}$

    Solution: Let $A=(0,1)$, $B=(1,0)$, $C=(0,-1)$, and $D=(-1,0)$. Vikas can go to points B, C, and D in any order, so there are $3!=6$ ways to choose the order in which to travel to those points. The minimum possible distance Vikas can travel between any two points is 2, and some of these ways will require Vikas to travel a straight path of length 2. We have three cases: Case 1: Vikas does not travel a straight path of length 2. There are two possibilities: $A-B-C-D-A$ and $A-D-C-B-A$. For each, there are 2 ways to travel between two adjacent points on the path. However, one way consists of going through the origin 4 times, so there are $2^4-1=15$ paths for each possibility. In total, there are 30 possible paths. Case 2: Vikas travels a straight path of length 2 two times. There are four possibilities: $A-B-D-C-A$, $A-C-B-D-A$, $A-C-D-B-A$, and $A-D-B-C-A$. For each, there are 2 ways to travel a bent path and 1 way to travel a straight path between two adjacent points. One way consists of going through the origin 4 times, so there are $2^2\cdot 1^2-1=3$ paths for each possibility. In total, there are 12 possible paths. Therefore, Vikas can take $30+12=42$ possible paths.

22. Answer: $\boxed{-27}$

Solution: Since the expression $((a+b)(b+c)(c+a))^3$ is symmetric, we attempt to arrange the expression to one that can be easily evaluated using Vieta's formulas: $((a+b)(b+c)(c+a))^3 = (a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+2abc)^3 = (a^2b+a^2c+abc+b^2a+b^2c+bac+c^2a+c^2b+cab-3abc+2abc)^3 = (a(ab+ac+bc)+b(ab+ac+bc)+c(ab+ac+bc)-abc)^3 = ((a+b+c)(ab+ac+bc)-abc)^3$. By Vieta's formulas, $a+b+c=1$, $ab+ac+bc=-5$, and $abc=-2$. Therefore, the expression becomes $(1\cdot -5-(-2))^3 = -27$.

23. Answer: $\boxed{\dfrac{1}{2}}$

Solution: Let the octahedron have side length s. By symmetry, the large tetrahedron must be regular. Then, since the octahedron is composed of four pairs of parallel faces, the four smaller tetrahedrons that form the difference in volume of the two solids must also be regular and have side length s. Then, the side length of the larger tetrahedron, formed by two side lengths of smaller tetrahedrons, has side length $2s$. The ratio of the volume of the smaller tetrahedrons to the larger tetrahedrons is $\frac{1}{2^3} = \frac{1}{8}$, so the ratio of the octahedron, which is the larger tetrahedron minus four smaller ones, to the larger tetrahedron is $\frac{1-4\cdot\frac{1}{8}}{1} = \frac{1}{2}$.

24. Answer: $\boxed{\dfrac{14}{81}}$

Solution: Since each move is equally likely, we can simply find the ratio of the number of desirable sequences of moves to the total number of sequences. Case 1: You move three times clockwise or counterclockwise and do nothing for a turn. The turn during which you stand still can be placed anywhere, for 4 choices. Then, the remaining three must be dedicated either to moving clockwise or counterclockwise, so we have a total of $4\cdot 2 = 8$ choices. Case 2: You move three times in one direction consecutively, and one time in the opposite direction. The one turn during which you move in the other direction can either be placed before or after the three turns. Furthermore, this turn can either be clockwise or counterclockwise, for a total of $2\cdot 2 = 4$ choices. Case 3: You move four times in the same direction. There are 2 ways in which this can happen. There are 3^4 total possible sequences, so the answer is $\frac{8+4+2}{3^4} = \frac{14}{81}$.

25. Answer: $\boxed{30}$

Solution: We consider sums mod 7:
For sums mod 0, we only need to add 7's, so all integers from 7 onwards with residue 0 can be expressed.
For sums mod 1, we first add two 11's and then add 7's, so all integers from 22 onwards with residue 1 can be expressed.
For sums mod 2, we first add $11+13+13$ and then add 7's, so all integers from 37 onwards with residue 2 can be expressed.
For sums mod 3, we first add $11+13$ and then add 7's, so all integers from 24 onwards with residue 3 can be expressed.
For sums mod 4, we first add 11 and then add 7's, so all integers from 11 onwards with residue 4 can be expressed.
For sums mod 5, we first add $13+13$ and then add 7's, so all integers from 26 onwards with residue 5 can be expressed.
For sums mod 6, we first add 13 and then add 7's, so all integers from 13 onwards with residue 6 can be expressed.
Summing up all the results (no pun intended), we have that all integers from 31 onwards can be expressed. This leaves 30 as the greatest integer that cannot be expressed.

26. Answer: $\boxed{4368}$

Solution: $12\binom{3}{3} + 11\binom{4}{3} + 10\binom{5}{3} + \cdots + 2\binom{13}{3} + \binom{14}{3} = [\binom{3}{3} + \binom{4}{3} + \cdots + \binom{14}{3}] + [\binom{3}{3} + \binom{4}{3} + \cdots + 2\binom{13}{3}] + [\binom{3}{3} + \binom{4}{3} + \cdots + 3\binom{12}{3}] + \cdots + [\binom{3}{3} + \binom{4}{3}] + [\binom{3}{3}]$. By the hockey-stick theorem, $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$. Applying the theorem to each sum in parentheses gives $[\binom{15}{4}] + [\binom{14}{4}] + [\binom{13}{4}] + \cdots + [\binom{5}{4}] + [\binom{4}{4}]$. Finally, applying the hockey-stick theorem on the total sum gives $\binom{16}{5} = 4368$.

27. Answer: $\boxed{\dfrac{720}{17}}$

Solution: Let the distance of the trip in one way be d. On the way there, Bob travels at an average of 45 mph, so he spends $\frac{d}{45}$ hours. On the way back, Bob spends $\frac{d/2}{30}$ hours half the distance and $\frac{d/2}{60}$ hours for the other half. Therefore, Bob traveled at an average rate of $\frac{2d}{\frac{d}{45}+\frac{d/2}{30}+\frac{d/2}{60}} = \frac{720}{17}$ mph.

28. Answer: $\boxed{\sqrt{166}}$

Solution: Since $BP \cdot PD = AP \cdot PC$, by the converse of Power of a Point, $ABCD$ is cyclic (which means that all of its vertices lie on some circle). Then, $\angle ABD \cong \angle ACD$ because the two angles intercept the same arc. By Angle-Angle similarity, $\triangle APB \sim \triangle DPC$, which means $\frac{AB}{BP} = \frac{DC}{CP} \Rightarrow DC = \frac{AB \cdot CP}{BP} = \frac{6 \cdot 3}{4} = \frac{9}{2}$. Similarly, $\triangle APD \sim \triangle BPC$, so $BC = \frac{AD \cdot CP}{DP} = \frac{3AD}{6} = \frac{1}{2}BC$. By Ptolemy's Theorem, $AB \cdot CD + BC \cdot AD = AC \cdot BD \Rightarrow 6 \cdot \frac{9}{2} + \frac{1}{2}AD \cdot AD = 11 \cdot 10 \Rightarrow \frac{1}{2}AD^2 = 83 \Rightarrow AD = \sqrt{166}$.

29. Answer: $\boxed{47}$

Solution: Let the square number be $(x+k)^2$, where k is a positive integer. Solving $x^2 + 17x + 17 = (x+k)^2$ for x, we get $x = \frac{k^2-17}{17-2k}$. If $k \leq 4$, $k^2 - 17 < 0$ and $17 - 2k > 0$, so $x < 0$. If $k \geq 9$, $k^2 - 17 > 0$ and $17 - 2k < 0$, so $x < 0$. Therefore, k can only equal 5, 6, 7, or 8. Checking these values, we find that only $k = 8$ yields an integer x. Therefore, $x = \frac{8^2-17}{17-2\cdot 8} = 47$ is the only solution.

30. Answer: $\boxed{\dfrac{1}{2}}$

Solution: Let p be the probability that Zach flips an odd number of heads for the nine unfair coins. Then, the probability that he flips heads for the fair coin to obtain an even number is $\frac{1}{2}p$. Similarly, the probability that he flips an even number of heads for the nine unfair coins and tails for the fair coin is $\frac{1}{2}(1-p)$. These two probabilities sum to $\frac{1}{2}$.

31. Answer: $\boxed{35}$

Solution: We note that $a_{n+1} = (a_n - 1)^2 + 1 \Rightarrow a_{n+k} = (a_n - 1)^{2^k} + 1$. Since $a_1 = 3$, we get $a_{1+k} = (a_1 - 1)^{2^k} + 1 \Rightarrow a_k = 2^{2^{k-1}} + 1$. Then, $a_1 a_2 \cdots a_{2012} = (2^{2^0}+1)(2^{2^1}+1)\cdots(2^{2^{2011}}+1) = (2^{2^0}-1)(2^{2^0}+1)(2^{2^1}+1)\cdots(2^{2^{2011}}+1)$. By repeatedly applying difference of squares, this expression collapses to $2^{2^{2012}} - 1$. Now, we proceed to find this number mod 100. It is known that $2^2 \equiv 2^{2+20k} \pmod{100}$ for integer k, which means that $2^{2012} \equiv 2^{12} \equiv 96 \pmod{100}$. Then, $2^{2012} \equiv 16 \pmod{20} \Rightarrow 2^{2^{2012}} \equiv 2^{16} \equiv 36 \pmod{100}$, so our final product has remainder 35.

32. Answer: $\boxed{\dfrac{20}{9}}$

Solution: Let $\triangle ABC$ have side length s. By Ptolemy's Theorem, $4s + 5s = s \cdot AP$, so $AP = 9$. $\angle APB$ intercepts the same arc as $\angle ACB$, so $m\angle APB = 60°$. Similarly, $\angle APC$ intercepts the same arc as $\angle ABC$, so $m\angle APC = 60°$. It follows that $\triangle ADB \sim \triangle ABP$ and $\triangle ADC \sim \triangle ACP$. Then, $\frac{AD}{DB} = \frac{AB}{BP} \Rightarrow AD = \frac{DB \cdot AB}{BP} = \frac{s}{5}DB$ and $\frac{AD}{DC} = \frac{AC}{AP} \Rightarrow AD = \frac{DC \cdot AC}{AP} = \frac{s}{4}(s - DB)$. Setting the two final equations equal to each other gives $\frac{s}{5}DB = \frac{s}{4}(s - DB) \Rightarrow DB = \frac{5}{9}s$. Since $\angle CAP$ and $\angle CBP$ intercept the same arc, they are congruent, which means $\triangle CAP \sim \triangle DBP$. Finally, from similar triangles, $\frac{CP}{CA} = \frac{DP}{DB} \Rightarrow DP = \frac{DB \cdot CP}{CA} = \frac{\frac{5}{9}s \cdot 4}{s} = \frac{20}{9}$.

33. Answer: $\boxed{\text{January 20}}$

Solution: We note that if a player can get the date to month $12 - x$ and day $31 - x$ for any valid x, then the player wins. This is because if the other player moves to month $12 - x + y$ and day $31 - x$ or to month $12 - x$ and day $31 - x + y$ for some positive y, then the first player can always bring the date to month $12 - x + y$, day $31 - x + y$, which is of the form described earlier. Eventually, $x = 0$, and the first player wins. Therefore, Hao must choose month $12 - 11$ and day $31 - 11$, or January 20.

34. Answer: $\boxed{45200000}$

Solution: Because Google says so. :)

35. Answer: $\boxed{34.1971}$

Solution: By trigonometry and calculator, it can be deduced that the side length of the pentagon is $\sqrt{\frac{1609.6}{\tan(54)}} \approx 34.1971$.

36. Answer: $\boxed{x \leq 8}$

Solution: The mean of all the valid numbers submitted was $8.\overline{36}$.