# 2nd Annual Lexington Mathematical Tournament
# Theme Round

Solutions

## 1 Interdisciplinary

1. Answer: $\boxed{13.8}$

   The absolute value of the error, as a fraction, is defined by Error $= \dfrac{|\text{Guess} - \text{Actual}|}{\text{Actual}}$. Since the guess by Bob was 100 years and the actual length of the war was 116 years, the error is $\dfrac{|100-116|}{116} \approx .138$. As a percent, this error is approximately 13.8%.

2. Answer: $\boxed{11}$

   Since the average length of the four poems not including the *Cypria* is 3.5, the total length of them is $3.5 \times 4 = 14$ books. The average length of all five poems is 5, so the total length of all poems is $5 \times 5 = 25$ books. Thus, the *Cypria* must be $25 - 14 = 11$ books long.

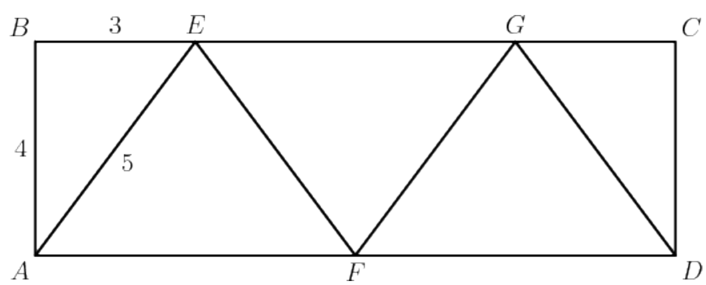
3. Answer: $\boxed{1/360}$

   First of all, Zaroug must actually pick the four noble gases correctly. Since he is picking four elements, there is only one way to select the four noble gases and $\binom{6}{2} = 15$ total ways to pick four elements. Thus, the probability of picking correctly is $1/15$. When Zaroug then goes to arrange the elements in order, only one such arrangement is the correct order. There are $4! = 24$ ways to arrange them, so the probability he gets the order right is $1/24$.

   The probability that Zaroug gets both right is then $(1/15)(1/24) = 1/360$.

4. Answer: $\boxed{7.5}$

   Because of perfect reflection and the fact that $BE/BC = 1/4$, we can draw in the path of the ball below.




   From here we can see that there are four paths $\overline{AE}, \overline{EF}, \overline{FG}, \overline{GD}$, all of which have the same length. We know that $BE = 12/4 = 3$, so by the Pythagorean theorem, $AE = \sqrt{3^2 + 4^2} = 5$. Thus, $EF = FG = GD = 5$ as well.

   Along the path $\overline{AE}$, the ball is traveling at 10 units per second, so it takes $5/10 = 0.5$ seconds to traverse this path. Along $\overline{EF}$, the ball is now traveling at half the speed, or 5 units per second, so it takes $5/5 = 1$ second here. Along $\overline{FG}$, the ball travels at $5/2 = 2.5$ units per second and takes $5/2.5 = 2$ seconds. Finally, along $\overline{GD}$, the ball travels at $2.5/2 = 1.25$ units per second and takes $5/1.25 = 4$ seconds to travel to $D$.

   Overall, it takes $0.5 + 1 + 2 + 4 = 7.5$ seconds.

5. Answer: $\boxed{9}$

We have

$$\begin{aligned}\frac{1}{10} = \frac{1}{m} + \frac{1}{n} \Rightarrow \frac{10mn}{10} &= \frac{10mn}{m} + \frac{10mn}{n} \\ &\Rightarrow mn = 10m + 10n \\ &\Rightarrow mn - 10m - 10n + 100 = 100 \\ &\Rightarrow (m-10)(n-10) = 100.\end{aligned}$$

Since $m$ and $n$ are positive integers, $m - 10$ must be some factor of 100 and $n - 10$ must be the corresponding factor so that they multiply to 100. Thus, the number of solutions $(m, n)$ is equal to the number of values of $m$ such that $m - 10$ is a factor of $100 = 2^2 \cdot 5^2$. There are $(2+1)(2+1) = 9$ factors of 100, so there are 9 ordered pairs $(m, n)$.

## 2 Schafkopf

6. Answer: $\boxed{8/13}$

There are four each of 7, 8, 9, 10, J, Q, K, A used in Schafkopf, for a total of 32 cards used in the game. Since there are 52 cards in a whole deck, the probability of drawing a Schafkopf card is $32/52 = 8/13$.

7. Answer: $\boxed{7/16}$

In each of the other three suits, there are six cards that are not part of the trump suit, so there are $3 \times 6 = 18$ non-trump Schafkopf cards. Since there are 32 total Schafkopf cards, $32 - 18 = 14$ cards are in the trump suit and the probability we desire is $14/32 = 7/16$.

8. Answer: $\boxed{3/248}$

This is equivalent to picking two cards out of 32 to be the blind. There are $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$ ways to have 2 queens in the blind and $\binom{32}{2} = \frac{32 \cdot 31}{2} = 496$ total ways to form a blind. The probability that the blind consists of 2 queens is then $6/496 = 3/248$.

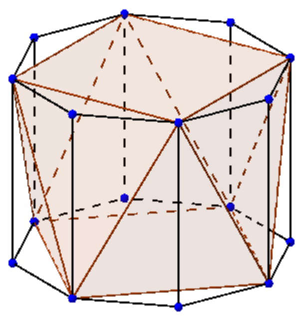
9. Answer: $\boxed{60}$

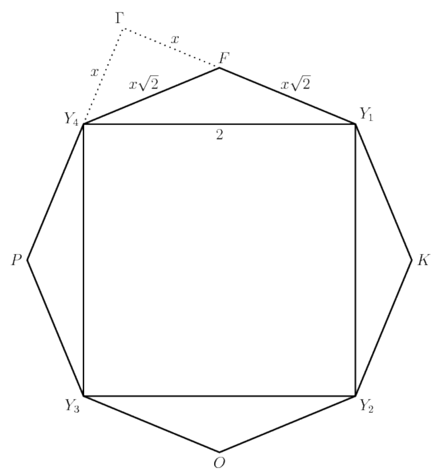
We first note that we can seat the five players in any order we want, so suppose we pick the order Peter, J., Leon, Ryan, Carl. Second, because the seats are numbered from 1 to 5, there is no rotational symmetry that can cause us to overcount.

There are 5 places that Peter could choose to sit in. After Peter sits down, J. would have 4 spots, but 2 of them are next to Peter and thus forbidden, so J. actually only has $4 - 2 = 2$ choices. Once J. sits down, we have no restrictions, so Leon has 3 choices, Ryan has 2 choices, and Carl's seat is forced. Thus, there are $5 \times 2 \times 3 \times 2 = 60$ ways for the players to sit.

10. Answer: $\boxed{\dfrac{16+8\sqrt{2}}{3}}$

To get the volume of the figure, we consider the octagonal prism of height 2 where the vertices of the bases are the feet of the perpendiculars of the vertices of polyhedron $S'C'H'A'FKOP$ onto the planes containing the original square bases of the polyhedron. The original points $S'C'H'A'$ then make up every other vertex of one regular octagon base, and the points $FKOP$ are the vertices of the other base that are not below the points $S'C'H'A'$ Suppose that one base is $X_1S'X_2C'X_3H'X_4A'$ and the other is $FY_1KY_2OY_3PY_4$. We then remove 8 triangular pyramids, where the heights connect corresponding vertices of the two bases and the triangular bases consist of three consecutive vertices of the octagonal bases, centered at points other than $X_i$ or $Y_i$.






To find the volume of the octagonal prism, we first find the area of the octagon base and then multiply by height. Consider base $FY_1KY_2OY_3PY_4$. We extend $Y_1F$ and $PY_4$ to meet at point $\Gamma$. Let $F\Gamma = x$. Since we are dealing with a regular octagon, it follows that $Y_4\Gamma F$ is an isosceles right triangle, so $FY_4 = x\sqrt{2} = FY_1$, $FY = AY$, and $Y_4\Gamma = x$. Furthermore, we note that since $Y_1Y_4$ is the projection of a side of square $S'C'H'A'$ by definition, $Y_1Y_4 = 2$. Applying the Pythagorean theorem to right triangle $Y_4\Gamma Y_1$, we have

$$x^2 + (x + x\sqrt{2})^2 = 2^2 \Rightarrow x^2 = 2 - \sqrt{2}.$$

Now, we can write the area of the octagon as

$$[FY_1KY_2OY_3PY_4] = [Y_4FY_1] + [Y_1KY_2] + [Y_2OY_3] + [Y_3PY_4] + [Y_1Y_2Y_3Y_4],$$

where the square brackets denote area. We note that the areas of the four triangles must all be equal in a regular octagon, so it suffices to compute $[Y_4FY_1]$. The base $FY_1$ is equal to $x\sqrt{2}$ from the previous part and the height is $Y_4\Gamma = x$, so

$$[Y_4FY_1] = \frac{1}{2}(x)(x\sqrt{2}) = \frac{x^2\sqrt{2}}{2}.$$

Because $Y_1Y_2Y_3Y_4$ is the projection of square $S'C'H'A'$ onto the plane of $KPOF$, $[Y_1Y_2Y_3Y_4] = 2^2 = 4$. Thus, the total area of the octagon is

$$[FY_1KY_2OY_3PY_4] = 4\left(\frac{x^2\sqrt{2}}{2}\right) + 4 = 2x^2\sqrt{2} + 4.$$

With a height of 2, the volume of the octagonal prism is

$$2(2x^2\sqrt{2} + 4) = 4x^2\sqrt{2} + 8.$$

Now we have to find the volume of each triangular pyramids. Consider the pyramid with base $Y_4FY_1$. The area of this base has been established in the previous part to be equal to $\frac{x^2\sqrt{2}}{2}$. Since the height is 2, the volume of the triangular pyramid is then

$$\frac{1}{3} \cdot 2 \cdot \frac{x^2\sqrt{2}}{2} = \frac{x^2\sqrt{2}}{3}.$$

There are 8 of these pyramids, therefore the total volume of the original polyhedron $S'C'H'A'FKOP$ is

$$\begin{aligned} 4x^2\sqrt{2} + 8 - \frac{8x^2\sqrt{2}}{3} &= \frac{24 + 4x^2\sqrt{2}}{3} \\ &= \frac{24 + 4(2-\sqrt{2})\sqrt{2}}{3} \\ &= \frac{16 + 8\sqrt{2}}{3}. \end{aligned}$$

# 3 Sophomore Eating Habits

11. Answer: $\boxed{5}$

    A large pie has an area of $4^2\pi = 16\pi$. A small pie has an area of $2^2\pi = 4\pi$. We see that 4 small pies have a total area of $4 \times (4\pi) = 16\pi$, which equals the area of a large pie, so in order for the small pies to total more than the large pie in area, we need $4 + 1 = 5$ small pies.

12. Answer: $\boxed{18}$

    Let's call the numbers of cookies and ramen packs sold on the first day $c$ and $r$ respectively. We can write the system of equations

    $$\begin{aligned} c + .5r &= 14 \\ 2c + 1.5r &= 37. \end{aligned}$$

    Solving this system of equations, we see that $.5r = 9$ and thus $r = 18$.

13. Answer: $\boxed{2011}$

    We know that $\frac{x}{2011}$ has to be around 2011, so $x$ has to around $2011^2$. Suppose that $x$ is equal to $2011^2 + y$ for some integer $y$, not necessarily positive. The number of pieces Surya makes is the closest integer to

    $$\frac{2011^2 + y}{2011} = 2011 + \frac{y}{2011}.$$

    In order for this value to be 2011, it must be the case that $-\frac{1}{2} \leq \frac{y}{2011} < \frac{1}{2}$. Otherwise, $2011 + \frac{y}{2011}$ will not round to 2011, by how rounding is defined. Therefore, we find that $-1005.5 \leq y < 1005.5$, so $y$ must be an integer from $-1005$ to 1005 inclusive, and all integers in this range work. The number of integers from $-1005$ to 1005 is $2(1005) + 1 = 2011$, and this is the number of solutions for $x$ as well.

14. Answer: $\boxed{30\sqrt{5}}$

Let the center of the pizza be $O$ and the tangency points on $\overline{AB}$ and $\overline{AD}$ be $E$ and $F$, respectively. Suppose $\overline{FE}$ intersects $\overline{OA}$ at $G$. Triangle $OFE$ is an isosceles triangle and $\overline{OA}$ is the perpendicular bisector of $\overline{FE}$. Thus, $GE = FE/2 = 3$. The area of the pizza is $45\pi$, so the radius $OE = \sqrt{45} = 3\sqrt{5}$. Now, note that since $ABCD$ is a rhombus, $\angle GOB$ is right, so

$$\begin{aligned}
m\angle GOE &= 90^\circ - m\angle BOE \\
m\angle BOE &= 90^\circ - m\angle OBE \\
m\angle GOE &= 90^\circ - (90^\circ - m\angle OBE) \\
m\angle GOE &= m\angle EBO.
\end{aligned}$$

We already know that $m\angle BEO = m\angle OGE = 90^\circ$. Thus, triangle $AEG$ is thus similar to triangle $OBE$. In particular,

$$\begin{aligned}
\frac{OE}{GE} &= \frac{OB}{OE} \\
\frac{3\sqrt{5}}{3} &= \frac{OB}{3\sqrt{5}} \\
OB &= 15.
\end{aligned}$$

Using the Pythagorean theorem,

$$BE = \sqrt{15^2 - (3\sqrt{5})^2} = 6\sqrt{5}.$$

We see that triangle $OBE$ is also similar to triangle $AOB$ as $m\angle OBE = m\angle OBA$ and $m\angle OEB = m\angle AOB = 90^\circ$ From this we have

$$\begin{aligned}
\frac{AB}{OB} &= \frac{OB}{BE} \\
\frac{AB}{15} &= \frac{15}{6\sqrt{5}} \\
AB &= \frac{15\sqrt{5}}{2}.
\end{aligned}$$

Since the box is a rhombus, the perimeter is $4(AB) = 30\sqrt{5}$.

15. Answer: $\boxed{1339}$

The last flower that Malik decorates will be the 2011th flower he decorates, since there are 2011 flowers to place. Since we started with the first flower in place 1, the third to last flower will be in the position $1 + 2008n$, where position $a$ is equivalent to position $a + 2011k$ for all integers $k$. In addition, because we know that this flower is in position 6, we have

$$\begin{aligned}
1 + 2008n &\equiv 6 \pmod{2011} \\
-3n &\equiv 5 \pmod{2011} \\
-3n &\equiv 2016 \pmod{2011} \\
n &\equiv -672 \equiv 1339 \pmod{2011}.
\end{aligned}$$

Since $1 \leq n \leq 2011$, $n = 1339$.