2nd Annual Lexington Mathematical Tournament
Team Round

Solutions

1 Potpourri

1. Answer: $\boxed{\sqrt{337}}$

   We have $AB = 9$ and $BC = 16$. Since the right angle is at B, the length of $\overline{AC}$, by the Pythagorean theorem, is $\sqrt{9^2 + 16^2} = \sqrt{337}$.

2. Answer: $\boxed{11}$

   Note: The answers to problem 2 of the individual round, problem 2 of the theme round, and problem 2 of this round are all 11.

   To maximize $m - n$ for integers m and n, we want m to be as positive as possible and n to be as negative as possible. Since $a^2 = (-a)^2$ for all a, we can find positive integer solutions (m, n) and then take the negative value of n. With guess and check, we get $(8, 1)$, $(7, 4)$, $(4, 7)$, and $(1, 8)$ as viable pairs. By making the aforementioned modifications, we evaluate $m - n$ for $(8, -1)$, $(7, -4)$, $(4, -7)$ and $(1, -8)$ to find that $7 - (-4)$ and $4 - (-7)$ produce the maximum value, 11.

3. Answer: $\boxed{2/5}$

   Since there are 9 problems and only 6 people, everyone will get at least 1, and then three will be left over. Thus, the three people that get 2 problems are the person who picks up the problems, the person to his or her left, and the person that is two spots to the left of the person who picks up the problems. Therefore, for Vishwesh to receive two problems, it must be the case that he picks up the problems, Aditya picks up the problems, or whoever sits to the right of Aditya picks up the problems. Since Aditya can't pick up the problems, there are 2 people that could have allowed for Vishwesh to get two problems. There are 5 people that could have picked up the problems in total, so the probability is $2/5$.

4. Answer: $\boxed{10:50}$

   Since the second pizza was 2 feet away from the fire the whole time, it took 15 minutes to bake. The first pizza must have then baked for 15 minutes, while 3 feet away from the fire, to completion. The time of baking is directly proportional to the distance from the fire, so if x is the amount of time it takes to bake while 3 feet away, $15/2 = x/3 \Rightarrow x = 22.5$ minutes. Because the rate of baking is even throughout the process, $15/22.5 = 2/3$ of the first pizza must have baked during this segment of time, which means $1/3$ of it must have baked during the first segment. Therefore, the first pizza baked 2 feet away from the fire for $15/3 = 5$ minutes. In total, the process took $5 + 15 = 20$ minutes, and 20 minutes after $10:30$ is $10:50$.

5. Answer: $\boxed{6}$

   We can use 6 coins to pay all possible sums from 10 to 20 inclusive with the set $\{4, 6, 7, 8, 9, 11\}$.

Sum	Combo
10	$4+6$
11	$4+7$
12	$4+8$
13	$4+9$
14	$6+8$
15	$6+9$
16	$7+9$
17	$8+9$
18	$7+11$
19	$8+11$
20	$9+11$

   We now must show that it is impossible to use 5 coins. Since we are forced to use exactly 2 coins, there are at most $\binom{5}{2} = 10$ different sums we can get, corresponding to the 10 different ways we can select two values for the coins. There are $20 - 10 + 1 = 11$ sums we have to be able to pay off, which is more than the amount of sums we can get off 5 coins. Thus, we cannot guarantee that Charon will be appeased.

6. Answer: $\boxed{31}$

   First, we note that whether the greatest common divisor of some integers is greater than 1 or not is entirely determined by the existence of prime factors, not how many times a prime factor appears in the factorization. Thus, we can assume, to minimize a, b, c, that each prime factor appears exactly once in the prime factorizations of a, b, c.

   We first determine that none of a, b, c can have exactly one prime factor. Without loss of generality, suppose that $a = p$ for some prime p. Since $\gcd(a, b) > 1$, we know that $p|b$. Similarly, $p|c$. Thus, p divides all three of a, b, c and $p|\gcd(a, b, c) = 1$, a contradiction.

   Now, suppose that a, b, c all have exactly two prime factors. In particular, let $a = pq$, where p and q are primes. Since $\gcd(a, b) > 1$, b must share some prime factor with a. Without loss of generality, let this be p. Similarly, c must share some prime factor with a. If this is p as well, then p divides all three of a, b, c, a contradiction, so $q|c$ instead. For some prime factors r and s, we have $b = pr$ and $c = qs$. Since $\gcd(b, c) > 1$, it must be the case that $r = s$, so

$$(a, b, c) = (pq, pr, qr)$$

   for some primes p, q, r. To minimize a, b, c and subsequently $a + b + c$, we let $p = 2$, $q = 3$, and $r = 5$. This gets us $a = 6$, $b = 10$, and $c = 15$, so $a + b + c = 31$.

7. Answer: $\boxed{7/2}$

   Since D is the midpoint of arc BC, angles BAD and CAD subtend the same arc length on the circle, so $m\angle BAX = m\angle BAD = m\angle CAD = m\angle CAD$ and $\overline{AX}$ is the angle bisector of $\angle BAC$. Let $\alpha = BX$ and $\beta = CX$, so $\alpha + \beta = 8$. By the Angle Bisector Theorem,

$$\frac{\alpha}{\beta} = \frac{7}{9} \Rightarrow 9\alpha = 7\beta.$$

   Solving the system of equations, we find that $\alpha = 7/2$.

8. Answer: $\boxed{47}$

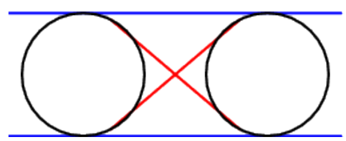
For all $n \le 10$, $2 \cdot 5 \cdot 10 = 100 | n!$ and the last two digits of $n!$ are 00. Thus, we only need to concern ourselves with $1!+3!+5!+7!+9!$. From here, we can just evaluate the last two digits of the factorials and then take the last two digits of the sum, to get $1+6+20+40+80 = 147 \to 47$.

9. Answer: $\boxed{66}$

To solve this problem, we note that what causes two terms to combine is that all variables appear as a factor in the terms the same number of times. Thus, we are not concerned with the coefficients of each term, but only how many times L, M, and T show up in each one. Let l, m, t be the number of times that L, M, T show up as a factor in a particular term, respectively. Since each factor of $L + M + T$ contributes 1 to exactly one of l, m, t, we must have $l + m + t = 10$. Furthermore, it is clear that l, m, t are nonnegative integers and that every triple (l, m, t) corresponds to a valid term in the expansion.

Our problem is then reduced to counting the number of ordered triples (l, m, t) of nonnegative integers with $l + m + t = 10$. By Balls and Urns, there are $\binom{10+3-1}{3-1} = 66$ ways to do this, so there are 66 terms in the simplified expansion.

10. Answer: $\boxed{0, \pm\frac{2\sqrt{5}}{5}}$



It is clear from the diagram that the two external tangents have slope 0. Thus, the solution focuses on the internal tangents.

Since the circles are congruent, by symmetry, the internal tangents pass through the midpoint of the segment connecting their centers, which is $(15/2, 0)$. If the slope of an internal tangent is m, then the point-slope form of this line is $y = m(x - 15/2)$. The equation of the circle centered at the origin is $x^2 + y^2 = 25$. Substituting our expression for y, we find

$$x^2 + m^2(x - 15/2)^2 = 25 \Rightarrow (4m^2 + 4)x^2 - 60m^2x + (225m^2 - 100) = 0.$$

For the line to be a tangent, this quadratic must only have one solution for x, which means the discriminant is 0. Therefore,

$$\begin{aligned}(60m^2)^2 - 4(4m^2 + 4)(225m^2 - 100) &= 0 \\ -2000m^2 + 1600 &= \\ m &= \pm\frac{2\sqrt{5}}{5}.\end{aligned}$$

2 Long Answer

2.1 Nested Radicals

1. A solution to part a is $(a, b) = (3, 5)$ and a solution to part b is $(a, b) = (6, 8)$. Since the two problems work out essentially the same way, we provide only the full solution to the second part.

   Squaring both sides of the equation, we get

   $$a + b + 2\sqrt{ab} = 14 + \sqrt{192}.$$

   Since a and b are integers, we know that $a+b$ is an integer. Without other definitive statements, to try to find <u>some</u> solution (a, b), we let $a + b = 14$ and $2\sqrt{ab} = \sqrt{192} \Rightarrow ab = 48$. Solving this system either by substituting to get a quadratic or by guess and check gets $(a, b) = (6, 8)$ as one solution.

   Note: Problem 2 definitely shows that it must be the case that $a + b = 14$ *and not any other value, but we cannot use it unless we wish to prove the result first as a lemma.*

2. For part a, we rearrange the equation given as $\sqrt{q} - \sqrt{b} = a - p$. Since a and p are integers, $a - p$ is equal to some integer n, so $\sqrt{q} - \sqrt{b} = n$. Solving for b, we find

   $$\sqrt{b} = \sqrt{q} - n \Rightarrow b = q + n^2 - 2n\sqrt{q}.$$

   Since b, q, and n^2 are all rational, it must be the case that $2n\sqrt{q}$ is rational. We know that q is not a square, so $\sqrt{q}$ is irrational and the only way for $2n\sqrt{q}$ to be rational is if $n = 0$. To see this, let $n \neq 0$ and suppose $2n\sqrt{q} = k$ for some rational number k. Then, $\sqrt{q} = k/2n$, and the right hand side is rational, so $\sqrt{q}$ would be rational, a contradiction.

   For part b, since $n = 0$, $\sqrt{q} - \sqrt{b} = 0 \Rightarrow \sqrt{q} = \sqrt{b}$, so $b = q$. Substituting this back into the original equation,

   $$a + \sqrt{b} = p + \sqrt{b} \Rightarrow a = p.$$

3. We square both sides like we did in problem 1 to get

   $$a + b + \sqrt{4ab} = p + \sqrt{q}.$$

   Since q is not a perfect square, by the result of problem 2,

   $$\begin{aligned} a + b &= p \\ 4ab &= q. \end{aligned}$$

   In particular, since a and b are integers, $q = 4ab$ is a multiple of 4, as desired.

4. We evaluate $p^2 - q$ as follows.

   $$\begin{aligned} p^2 - q &= (a + b)^2 - 4ab \\ &= a^2 + 2ab + b^2 - 4ab \\ &= a^2 - 2ab + b^2 \\ &= (a - b)^2. \end{aligned}$$

   Since a and b are integers, $a - b$ is an integer and thus $p^2 - q$ is a perfect square.

2.2 Kinematics

5. For average velocity v and time t, we have $d = vt$. We can write

$$\begin{aligned} v &= \frac{1}{2}(v_t + v_0) \\ &= \frac{1}{2}((v_t - v_0) + 2v_0). \end{aligned}$$

Since acceleration, by definition, is given by $a = \dfrac{v_t - v_0}{t}$, we have $at = v_t - v_0$ and

$$v = \frac{1}{2}(at + 2v_0) = \frac{1}{2}at + v_0.$$

Substituting this into the first equation for d,

$$d = \frac{1}{2}at^2 + v_0 t.$$

6. We make use of the three equations

$$\begin{aligned} a &= \frac{v_t - v_0}{t} \\ v &= \frac{v_t + v_0}{2} \\ d &= vt = \frac{t(v_t + v_0)}{2}. \end{aligned}$$

Substituting, we get

$$\begin{aligned} 2ad &= 2 \cdot \frac{v_t - v_0}{t} \cdot \frac{t(v_t + v_0)}{2} \\ &= (v_t - v_0)(v_t + v_0) \\ &= v_t^2 - v_0^2. \end{aligned}$$

7. In this specific scenario, our initial velocity is $v_0 = 20$ meters per second and the acceleration is -10 meters per second squared. From part 5, at time t, we have

$$d = -5t^2 + 20t.$$

In order for the ball to hit the ground again, it must be the case that $d = 0$. Substituting and solving, we find $t = 0$ or $t = 4$. Since $t = 0$ is when the ball is shot up, $\boxed{t = 4}$ must be when the ball comes down.

From the result of problem 6, we have $v_t^2 - v_0^2 = 2ad$. When the ball hits the ground, $d = 0$, and since $v_0 = 20$, we have

$$v_t^2 - 20^2 = 0 \Rightarrow v_t = \pm 20.$$

Since the ball is coming down when it hits the ground, $v_t < 0$ and so $\boxed{v_t = -20}$ meters per second.