2nd Annual Lexington Mathematical Tournament
Guts Round

Solutions

1. Answer: $\boxed{-104}$

$$\begin{aligned}(1-2(3-4(5-6)))(7-(8-9)) &= (1-2(3-4(-1)))(7-(-1))\\ &= (1-2(7))(8)\\ &= (-13)(8) = -104.\end{aligned}$$

2. Answer: $\boxed{70}$

   Subtracting 19 from every element in the set, we get the new set $\{1, 2, 3, ..., 69, 70\}$. Since we did nothing to change the number of elements in the set, the number of elements in the original set is clearly 70.

3. Answer: $\boxed{105}$.

   Let A be the ~~droid~~ angle we're looking for. By the definitions of complement and supplement, we have

$$\begin{aligned}3(90-(180-A)) &= A-60\\ 3A-270 &= A-60\\ A &= 105.\end{aligned}$$

4. Answer: $\boxed{-6.4}$

   A change of $x\%$, where x is negative for a percent decrease and positive for a percent increase, is equivalent to multiplying by a factor of $1+\frac{x}{100}$. Thus, our three percent changes are equivalent to multiplying by $\frac{9}{10}\cdot\frac{4}{5}\cdot\frac{13}{10} = .936$. By using our equivalence again and setting c to be the overall percent change, we have

$$1+\frac{c}{100} = .936 \Rightarrow c = -6.4.$$

5. Answer: $\boxed{44}$

   Consider base L with endpoints $(2, 3)$ and $(13, 3)$. Since the $y-$coordinates are equal, L is parallel to the $x-$axis. Thus, the perpendicular from $(8, 11)$ has length $11-3 = 8$. The length of L is just $13-2 = 11$, so the area of the triangle we desire is $11\times 8/2 = 44$.

6. Answer: $\boxed{1/36}$

   From the first bin, the pen picked must be blue, since there are no pens of other colors. From the second bin, in order to have all three colors represented, the pen picked must be green. There are 2 green pens and $2+4 = 6$ pens total in this bin, so the probability of a green pen being picked is $2/6 = 1/3$. Similarly, we must pick a red pen from the third bin, which has probability $1/(1+5+6) = 1/12$. All of these things are independent and must happen, so the probability we desire is $1(1/3)(1/12) = 1/36$.

7. Answer: $\boxed{12}$

   We factor the equation as $(a+b)(a-b) = 23$. Since $a, b > 0$, $a+b > 0$, and so $a-b > 0$ as well. Furthermore, since a and b are integers, $a+b$ and $a-b$ are integers that multiply to 23. As it so happens, 23 is prime, so $a+b = 23$ and $a-b = 1$. Solving, we find that $a = 12$ and $b = 11$.

8. Answer: $\boxed{2^3 \cdot 3^1 \cdot 5^2 \cdot 7^4}$

   For positive integers a and b, let p be a prime number and let a' and b' be positive integers not divisible by p such that $a = p^x a'$ and $b = p^y b'$ for nonnegative integers x and y. Let g be the greatest common divisor of a and b. Clearly, $p^{\min\{x,y\}}$ divides both a and b, and so $p^{\min\{x,y\}}|g$. In addition, $p^{\min\{x,y\}+1}$ will not divide one of a and b by having an extra factor of p not present in a or b, so $p^{\min\{x,y\}}$ is, in fact, the largest power of p that divides g. In other words, $p^{\min\{x,y\}}$ is the power of p that appears in the prime factorization of g.

   Running through this process for our specific problem and $p = 2, 3, 5, 7$, we find the greatest common divisor has prime factoriation $2^3 \cdot 3^1 \cdot 5^2 \cdot 7^4$ by taking the minimal exponents that show up for each prime factor.

9. Answer: $\boxed{3}$

   We notice that there is cyclic symmetry in the coefficients of the terms in each of the equations. This motivates us to consider adding the three equations. Doing exactly this, we get

$$6a + 6b + 6c = 6 \Rightarrow 3a + 3b + 3c = 6/2 = 3.$$

10. Answer: $\boxed{21}$

    The number 11 is prime, so any positive integer dividing 11^{20} must only contain prime factors of 11. Thus, it can range from 11^0 to 11^{20}, for a total of $20 - 0 + 1 = 21$ values.

11. Answer: $\boxed{7}$

    Writing $64 = 2^6$, we have

$$\begin{aligned} 2^{x-5} &= (2^6)^{x/\alpha} \\ 2^{x-5} &= 2^{6x/\alpha} \\ x - 5 &= 6x/\alpha \\ x &= \frac{5}{1 - 6/\alpha}. \end{aligned}$$

    Upon finding the answer $\alpha = 21$, we can evaluate x to be $\dfrac{5}{1 - 6/21} = 7$.

12. Answer: $\boxed{25/2}$

    Since LMT is a right triangle with hypotenuse $\overline{LT}$ and Z is the midpoint of $\overline{LT}$, Z is also the intersection of the perpendicular bisectors of $\overline{LM}$ and $\overline{MT}$. Thus, $ZL = ZM = ZT$. From this, we find $ZL + ZT = LT = 2(ZM)$, but we also have $LT = 4\beta - 3$, so

$$ZM = \frac{4\beta - 3}{2}.$$

    Upon finding the answer $\beta = 7$, we can evaluate ZM to be $\dfrac{4(7) - 3}{2} = \dfrac{25}{2}$.

13. Answer: $\boxed{12/7}$

We can just sum as

$$\begin{aligned}\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21} &= \frac{4}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}\\ &= \frac{3}{2}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}\\ &= \frac{8}{5}+\frac{1}{15}+\frac{1}{21}\\ &= \frac{5}{3}+\frac{1}{21}\\ &= \frac{12}{7}.\end{aligned}$$

For a more general solution, note that if T_n is the nth triangular number, then our sum is a special case of evaluating

$$\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_n}.$$

Writing $T_n = \dfrac{n(n+1)}{2}$ and $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, we have

$$\begin{aligned}\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_n} &= \frac{2}{1(2)}+\frac{2}{2(2)}+\cdots+\frac{2}{n(n+1)}\\ &= \frac{2}{1}-\frac{2}{2}+\frac{2}{2}-\frac{2}{3}+\cdots-\frac{2}{n}+\frac{2}{n}-\frac{2}{n+1}\\ &= 2-\frac{2}{n+1} = \frac{2n}{n+1}\end{aligned}$$

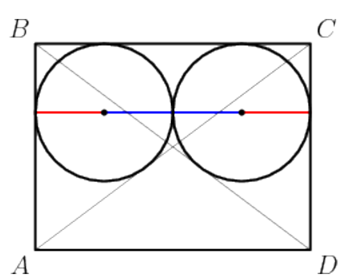
Our particular problem is $n=6$, for which we get $2(6)/(6+1) = 12/7$.

14. Answer: $\boxed{7/10}$

It is possible to solve this system explicitly, either by substitution and solving a single-variable quadratic or by guess and check, to get $x=2$ and $y=5$ or vice versa. With this, the sum of the reciprocals is $1/2+1/5 = 7/10$.

For a method that doesn't require finding x and y directly, we write $\frac{1}{x}+\frac{1}{y} = \frac{x+y}{xy}$. We already know $x+y=7$, so it remains to find xy. To do this, we note $(x+y)^2-(x^2+y^2) = 2xy$, so $2xy = 7^2-29 \Rightarrow xy = 10$. Then, $\dfrac{x+y}{xy} = \dfrac{7}{10}$.

15. Answer: $\boxed{2}$



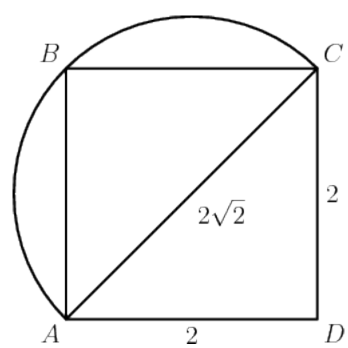
Let r be the radius of the inscribed circles, since it is clear that they are congruent. From the diagram, it is evident that the distance between the centers is equal to $4-2r$, so it suffices to find r. We use the formula $A = rs$, where $A = 3(4)/2 = 6$ is area and $s = (3+4+5)/2 = 6$ is semiperimeter, to find that $6 = 6r \Rightarrow r = 1$. Thus, the distance we wish to find is $4-2r = 2$.

16. Answer: $\boxed{5049}$

We note for all positive integers n that $\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1$. Thus, our sum is equal to

$$\begin{aligned} 2+3+4+\cdots+99+100 &= (1+2+3+\cdots+99+100) - 1 \\ &= \frac{100 \cdot 101}{2} - 1 \\ &= 5049. \end{aligned}$$

17. Answer: $\boxed{\pi - 2}$



The area of the region is equal to the area of the semicircle minus the area of triangle ABC. The diameter of the semicircle has length equal to $AC = 2\sqrt{2}$, since the diagonal of a square is $\sqrt{2}$ of the side length. Thus, the radius is $\sqrt{2}$ and the area of the semicircle is $\frac{\pi(\sqrt{2})^2}{2} = \pi$. The area of triangle ABC is $\frac{2 \cdot 2}{2} = 2$, so the desired region has area $\pi - 2$.

18. Answer: $\boxed{8}$

We can simplify as $\frac{37k-30}{k} = 37 - \frac{30}{k}$. Clearly, if $30/k$ is an integer, it is at most 30, so $37 - \frac{30}{k} > 37 - 30 = 7$ and is positive, so we just need $30/k$ to be an integer. The number of values of k for which this holds is precisely the number of divisors of $30 = 2^1 \cdot 3^1 \cdot 5^1$, which is $(1+1)(1+1)(1+1) = 8$ (to see this, use the logic of problem 10 for each prime factor).

19. Answer: $\boxed{27}$

The cross sections are circles of radii $\sqrt{576} = 24$ and $\sqrt{225} = 15$. To find the distance between the parallel planes, we will instead consider the intermediate distances from the center of the sphere to the planes. The sum or difference of these distances will then be the distance between the planes.

For a circular cross section of radius r in a sphere of radius R, let O be the center of the sphere, A be the center of the circle, and B be any point on the circle. It is clear that triangle OAB is right with right angle at A, so

$$(OA)^2 + (AB)^2 = (OB)^2 \Rightarrow OA = \sqrt{R^2 - r^2}.$$

This is the distance from O to the plane passing through the circular cross section. For our specific case, we can find the two distances desired to be $\sqrt{25^2 - 24^2} = 7$ and $\sqrt{25^2 - 15^2} = 20$. Thus, the maximal possible distance between the two planes is $20 + 7 = 27$.

20. Answer: $\boxed{2}$

    Suppose we can get some three consecutive integers k, $k+1$, $k+2$ from linear combinations of 3, 4, and 5. Then, by adding multiples of 3, we can get all positive integers $k+3n$, $k+3n+1$, and $k+3n+2$ for any nonnegative integer n. Collectively, $3n$, $3n+1$, and $3n+2$ range over all nonnegative integers, so we can thus generate all positive integers above k. Our problem is then finding the smallest set of three consecutive integers that can be generated. By letting one of the three variables be 1 and the other two be 0, we can see that it is easy to generate $3, 4, 5$. Hence, we can get all positive integers at least 3, so the only candidates for integers that cannot be obtained are 1 and 2. Obviously neither of these two can be generated, so we have 2 such positive integers.

21. Answer: $\boxed{3}$

    To construct a case where there is someone who was fooled by 3 people but nobody was fooled by 4 people, suppose 13 of the 14 fooled people were fooled by exactly 2 classmates, and then the last one was fooled by 3. This satisfies the given requirements, so we cannot guarantee anything at least 4.

    Suppose that each person that was fooled was fooled by at most 2 people. Then, the number of people that could have fooled someone else is at most $2 \times 14 = 28 < 29$, a contradiction. Thus, we guarantee that someone must have been fooled by at least 3 people.

22. Answer: $\boxed{6}$

    If we substitute our expression in for S, we see that we end up with exactly the same infinite continued fraction that defines S in the first place. Thus,

    $$S = 4 + \frac{12}{S} \Rightarrow S = 6, -2.$$

    Since clearly S must be positive in its definition, $S = 6$.

23. Answer: $\boxed{19}$

    Let b be the number of bananagram sets he bought and f be the number of flip-flops he bought. Then, $11b + 17f = 227$. Taking this equation modulo 11, we have

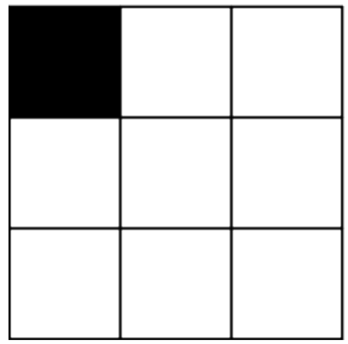
    $$6f \equiv 7 \pmod{11} \Rightarrow f \equiv 3 \pmod{11}.$$

    Thus, $f = 11k + 3$ for some integer k. Substituting, we find that $b = 16 - 17k$. For $k \leq -1$, $f \leq -11 + 3 = -8 < 0$, an impossibility since f is a positive integer. Similarly, if $k \geq 1$, $b \leq 16 - 17 = -1 < 0$, contradicting the fact that b is a positive integer. Therefore, $k = 0$, $f = 3$, $b = 16$, and the total number of items Jonathan bought is $f + b = 19$.

24. Answer: $\boxed{2}$

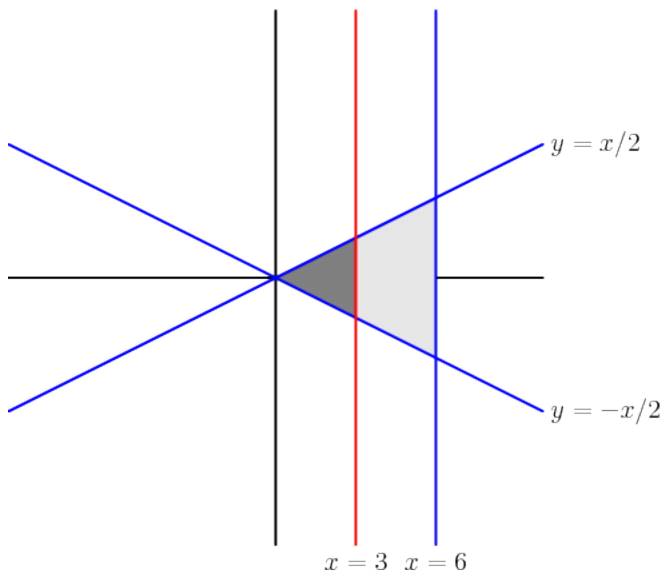
    If the center square is removed, then no other square could have been removed or else at least two squares along the outside will have at most one neighbor. Thus, the case where the center square is removed produces a maximal answer of 1. Now, we consider the case where the center square remains on the board.

    We first note that for the first move, we cannot remove an edge piece because then the two adjacent corner pieces will be lonely. Thus, we must remove a corner piece first.



Now, we note that we still cannot remove any edge pieces, and we also cannot remove one of the two corner pieces closest to the one already gone, because then the edge piece in the middle would be lonely. Hence, we can only remove the piece in the opposite corner, and it can quickly be seen that no more pieces can be removed. The maximal number of pieces that can be deleted is thus 2.

25. Answer: $\boxed{1/4}$



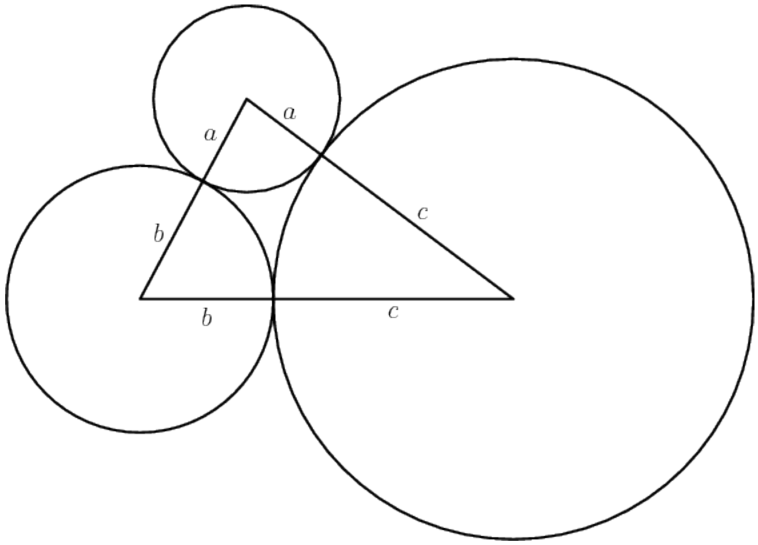
Since the maximal possible value of $|y| = x/2$ increases as x increases, it is clear that as we shift a point 3 units to the left, the point cannot be above or below the region of set $\mathcal{S}$. Therefore, we only concern ourselves with the $x-$coordinate of point $P(x, y)$.

In order for P' to lie within the set $\mathcal{S}$, it must be the case that

$$x + 3 \leq 6 \Rightarrow x \leq 3$$

This creates a new region $\mathcal{R}$ of dimensions $3/6 = 1/2$ that of the set $\mathcal{S}$. The probability we desire is then the ratio of the area of $\mathcal{R}$ to the area of $\mathcal{S}$, which is $(1/2)^2 = 1/4$.

26. Answer: $\boxed{473\pi}$



If the radii of the circles are $a < b < c$, then we have

$$\begin{aligned} a+b &= 17 \\ a+c &= 25 \\ b+c &= 28. \end{aligned}$$

To solve this system of equations, we add them up to get

$$2a+2b+2c = 70 \Rightarrow a+b+c = 35.$$

It directly follows that $a = 7$, $b = 10$, $c = 18$. The total area of the circles is $a^2\pi + b^2\pi + c^2\pi = 473\pi$.

27. Answer: $\boxed{(0,3),(1,2),(1,4),(2,3)}$

From the equation given,

$$\begin{aligned} x^2 - 2x + y^2 - 6y &= -9 \\ x^2 - 2x + 1 + y^2 - 6y + 9 &= -9 + 1 + 9 \\ (x-1)^2 + (y-3)^2 &= 1. \end{aligned}$$

Since x and y are integers, it follows that $(x-1)^2$ and $(y-3)^2$ are perfect squares, so one of them is 1 and the other is 0. If $(x-1)^2 = 0$ and $(y-3)^2 = 1$, then we have $x = 1$ and $y - 3 = \pm - 1 \Rightarrow y = 2, 4$. If $(x-1)^2 = 1$ and $(y-3)^2 = 0$, then $x - 1 = \pm 1 \Rightarrow x = 0, 2$ and $y = 3$. The four ordered pair solutions are $(0,3),(1,2),(1,4),(2,3)$.

28. Answer: $\boxed{30240}$

To count the number of desired arrangements, we will count the number of orderings where the A and O are together, and then subtract the number of orderings where the A and O are together and the two F's are together. Let $\mathcal{A}$ be the block of the two letters A and O. Then, we must arrange the eight letters in the "word" SCH$\mathcal{A}$FKPF, of which there are $8!/2 = 20160$ ways. In addition, there are 2 ways to arrange the two constituent letters of $\mathcal{A}$, for $20160 \times 2 = 40320$ ways. For counting when the two F's are together, let $\mathcal{F}$ be the block containing the F's. We then have the seven letters of SCH$\mathcal{A}\mathcal{F}$KP to arrange, as well as the two letters in $\mathcal{A}$. This leads to $7! \times 2 = 10080$ ways we need to subtract, so $40320 - 10080 = 30240$ arrangements are desired.

29. Answer: $\boxed{31186}$

From the recursion given, we establish

$$\begin{aligned} a_{2011} + a_{2010} &= a_{2009} + a_{2008} \\ &= a_{2007} + a_{2006} \\ &= a_{2005} + a_{2004} \\ &= \vdots \\ &= a_3 + a_2 \\ &= a_1 + a_0. \end{aligned}$$

Now, our sum is

$$\begin{aligned} a_0 + a_1 + a_2 + \cdots + a_{2010} + a_{2011} &= (a_0 + a_1) + (a_2 + a_3) + \cdots + (a_{2010} + a_{2011}) \\ &= 1006(a_0 + a_1) \\ &= 1006(20 + 11) = 31186. \end{aligned}$$

30. Answer: $\boxed{16}$

We first note that since 9 is a digit in base b, $b \geq 10$. It is also clear to see that for all b, $190_b = b^2 + 9b < b^2 + 10b + 25 = (b+5)^2$. Thus, we will look for when $190_b > (b+4)^2$, because then 190_b lies between two consecutive perfect squares and cannot be a perfect square. This occurs when

$$b^2 + 9b > b^2 + 8b + 16 \Rightarrow b > 16.$$

In order for 190_b to be a square, we must therefore have $b = 10, 11, 12, 13, 14, 15, 16$. Testing these seven numbers, we see that 16 is the only value that works.

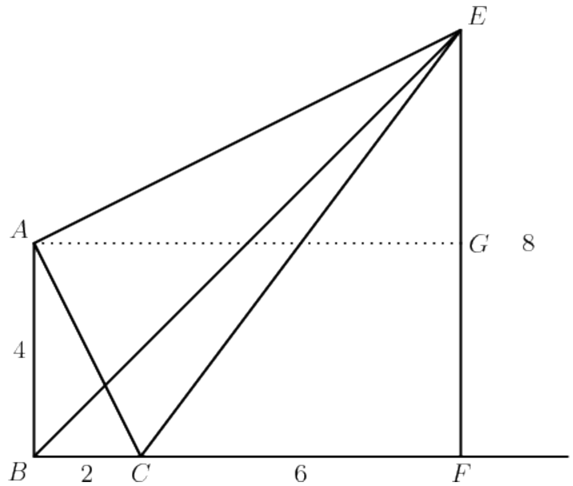
31. Answer: $\boxed{0, \pm\frac{1}{4}}$

Let $a = 4x - 1$ and $b = 4x + 1$. Cubing both sides of the equation, we get

$$\begin{aligned} (\sqrt[3]{a} + \sqrt[3]{b})^3 &= (\sqrt[3]{a+b})^3 \\ a + b + 3\sqrt[3]{a^2 b} + 3\sqrt[3]{ab^2} &= a + b \\ 3\sqrt[3]{a}\sqrt[3]{b}(\sqrt[3]{a} + \sqrt[3]{b}) &= 0 \\ 3\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{a+b} &= 0. \end{aligned}$$

Thus, we have $4x - 1 = 0$, $4x + 1 = 0$, or $8x = 0$, which gives us the three solutions $1/4$, $-1/4$, and 0.

32. Answer: $\boxed{20}$



Since E lies on the angle bisector of $\angle ABC = \angle ABF$, $m\angle EBF = 45°$. Since $\overline{EF} \perp \overline{FB}$, EFB is an isosceles right triangle with right angle at F. Thus, $FB = BE = 8$. We know that $BC = 2$, so $CF = 8 - 2 = 6$. By the Pythagorean Theorem,

$$\begin{aligned} EC &= \sqrt{6^2 + 8^2} \\ &= 10 \\ AC &= \sqrt{2^2 + 4^2} \\ &= 2\sqrt{5}. \end{aligned}$$

From here, we have two ways to proceed.

Method 1: We note that by definition, $m\angle BCA = m\angle ACE$. In addition, we observe that $CB/CA = CA/CE = 1/\sqrt{5}$. Therefore, by SAS similarity, $\triangle ABC \sim \triangle EAC$. The area of triangle ABC is $4 \cdot 2/2 = 4$, so the area of triangle ACE is

$$4 \cdot (\sqrt{5})^2 = 20.$$

Method 2: Draw perpendicular $\overline{AG}$ to $\overline{EF}$ with G on $\overline{EF}$. We can then find that $AG = 8$, $GF = 4$, and $EF = 4$. By the Pythagorean Theorem, $AE = \sqrt{4^2 + 8^2} = 4\sqrt{5}$. To find the area of ACE now that we have all the side lengths, we note that $(AC)^2 + (AE)^2 = (CE)^2$. Thus, triangle ACE is right with right angle at A and the area is

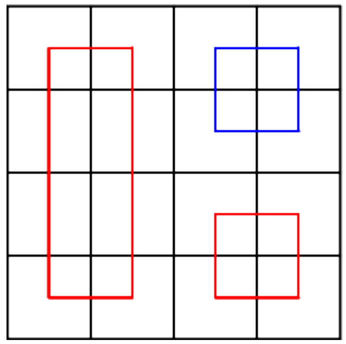
$$\frac{2\sqrt{5} \cdot 4\sqrt{5}}{2} = 20.$$

If we aren't able to find this, we can use Heron's formula with $s = 3\sqrt{5} + 5$ to get the area of

$$\begin{aligned} A &= \sqrt{(3\sqrt{5} + 5)(3\sqrt{5} - 5)(5 + \sqrt{5})(5 - \sqrt{5})} \\ &= \sqrt{(45 - 25)(25 - 5)} \\ &= 20. \end{aligned}$$

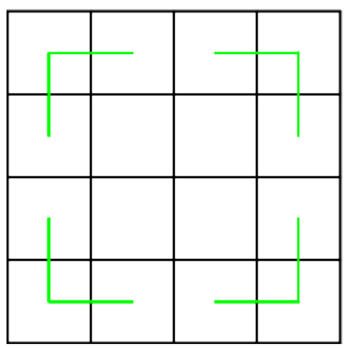
33. Answer: $\boxed{44}$

For each square, we must have one soul going in and one soul going out. We can therefore trace the paths of the souls as a set of directed cycles, one example of which is shown below. In the diagrams, blue indicates a clockwise rotation of souls and red indicates a counterclockwise rotation.



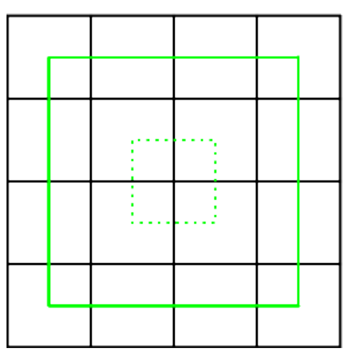
Our task is then to count the number of sets of directed cycles we can draw in the 4×4 grid, by counting the number of undirected cycles and then coloring each cycle blue or red corresponding to the direction of rotation. The cycle containing the bottom left square must be red, since the bottom left soul moves to the right.

Consider one of the four corners. Since a path goes in and then goes out in a different direction, the path must round the corner and we can generate partial cycles below.



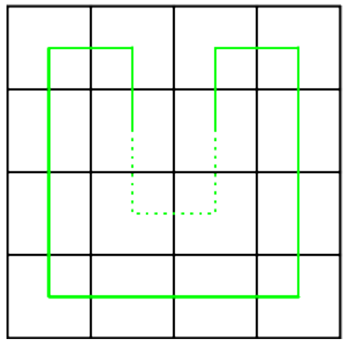
At each of these edges, the paths can then either turn in towards the center or join along the edge. We base our cases on how many of these paths are joined.

Case 1: All four paths along the edges are joined.



In this case, we must then have a small cycle inside around the 2×2. The outer loop must be colored red and the inside loop has a choice, so this gives us 2 possibilities.

Case 2: Exactly three paths along the edges are joined.

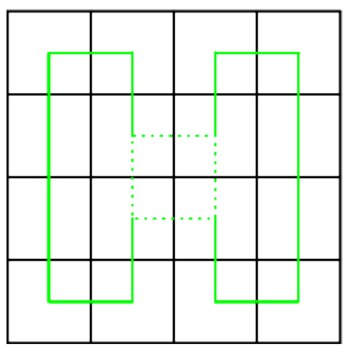


The two paths that turn in cannot immediately join up, because then we only have two squares left and we cannot have the two souls there directly switching. Thus, our path must be one contiguous cycle. There are four ways to rotate this cycle to get a new one, and the direction is fixed, so this case gives us 4 solutions.

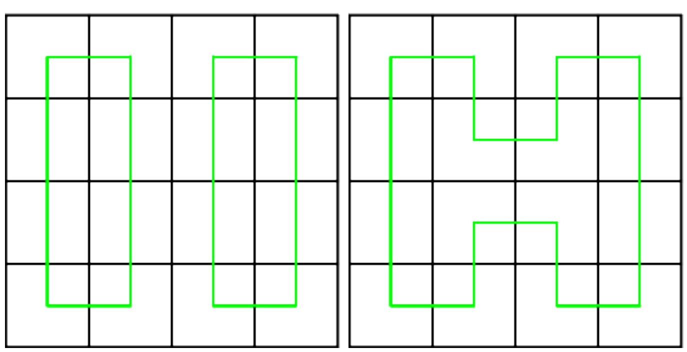
Case 3: Exactly two paths along the edges are joined.

Here, we have two subcases based on whether the two that are joined are adjacent to or opposite each other.

Subcase 3-1: The two joined paths are opposite each other.

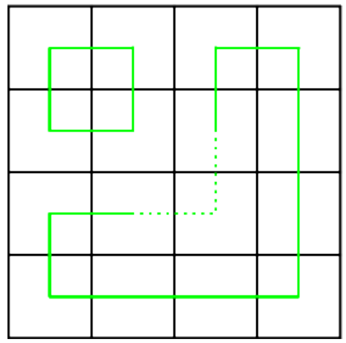


To finish off the pathing, we can either create two separate rectangular loops or we can create one I-shaped loop.



For the case on the left, we can rotate this to get 2 different possible sets, and the direction in one loop is fixed while the other has a choice, giving us $2 \times 2 = 4$ possible sets here. For the one on the right, we have 2 rotations and no choice of direction, for 2 sets here. There are thus $4 + 2 = 6$ solutions in this subcase.

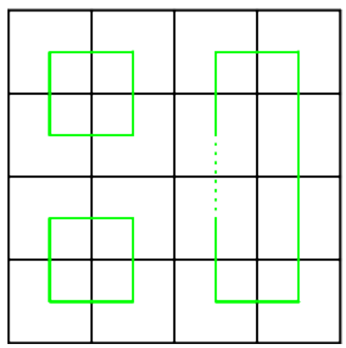
Subcase 3-2: The two joined paths are adjacent to each other.



The two free ends clearly only have one way to go, and then meet up. This creates two loops and four rotations, for $2 \times 4 = 8$ sets in this subcase.

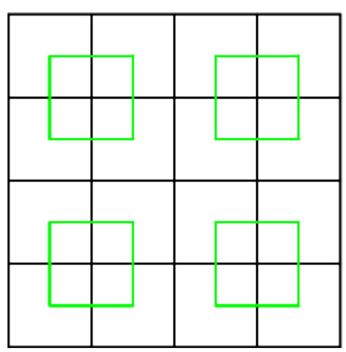
In the entire case, we have $6 + 8 = 14$ solutions.

Case 4: Exactly one path along the edges is joined.



It is quite evident from the diagram that what we end up with are two small square loops and one larger rectangular loops. There are 2 loops with free choice for direction, one without, and four rotations for $2^2 \times 4 = 16$ solutions.

Case 5: All paths along the edge go inward.



We end up with four square loops, giving us no rotations other than the identity, but three loops with free directional choice. This equates to $2^3 = 8$ sets of directed cycles.

In all, we have $2 + 4 + 14 + 16 + 8 = 44$ such ways in which the situation described can occur.

34. Answer: $\boxed{15919}$

The formula for a primitive Pythagorean triple $(m^2 - n^2, 2mn, m^2 + n^2)$ can be used to give a somewhat crude estimate by estimating the number of ordered pairs (m, n) with $m^2 + n^2 < 100000$. Further restricting this count with parity and common divisors gets a closer estimate and, to some extent, makes the numbers more workable.

A result proven by Lehmer in 1900 shows that as N increases to infinity, the number of primitive Pythagorean triples with hypotenuse less than N approaches $N/2\pi$. Some rather careful division gets an approximation of 15915, which is only 4 off from the true value. As far as the author of this problem knows, the only way to get the exact answer would be to brute force with a program.

35. Answer: $\boxed{1525}$

All that can be said about this problem is that apparently, there are surprisingly many words with L, M, T in order.

36. Answer: $\boxed{???}$

The guessed integers that resulted in positive score are listed on the LMT website.