2nd Annual Lexington Mathematical Tournament
Team Round

April 2, 2011

1 Potpourri [70]

1. Triangle ABC has side lengths $AB = 3^2$ and $BC = 4^2$. Given that $\angle ABC$ is a right angle, determine the length of $\overline{AC}$.

2. Suppose m and n are integers such that $m^2+n^2 = 65$. Find the largest possible value of $m-n$.

3. Six middle school students are sitting in a circle, facing inwards, and doing math problems. There is a stack of nine math problems. A random student picks up the stack and, beginning with himself and proceeding clockwise around the circle, gives one problem to each student in order until the pile is exhausted. Aditya falls asleep and is therefore not the student who picks up the pile, although he still receives problem(s) in turn. If every other student is equally likely to have picked up the stack of problems and Vishwesh is sitting directly to Aditya's left, what is the probability that Vishwesh receives exactly two problems?

4. Paul bakes a pizza in 15 minutes if he places it 2 feet from the fire. The time the pizza takes to bake is directly proportional to the distance it is from the fire and the rate at which the pizza bakes is constant whenever the distance isn't changed. Paul puts a pizza 2 feet from the fire at 10:30. Later, he makes another pizza, puts it 2 feet away from the fire, and moves the first pizza to a distance of 3 feet away from the fire instantly. If both pizzas finish baking at the same time, at what time are they both done?

5. You have n coins that are each worth a distinct, positive integer amount of cents. To hitch a ride with Charon, you must pay some unspecified integer amount between 10 and 20 cents inclusive, and Charon wants exact change paid with exactly two coins. What is the least possible value of n such that you can be certain of appeasing Charon?

6. Let a, b, and c be positive integers such that $\gcd(a, b)$, $\gcd(b, c)$ and $\gcd(c, a)$ are all greater than 1, but $\gcd(a, b, c) = 1$. Find the minimum possible value of $a + b + c$.

7. Let ABC be a triangle inscribed in a circle with $AB = 7$, $AC = 9$, and $BC = 8$. Suppose D is the midpoint of minor arc BC and that X is the intersection of $\overline{AD}$ and $\overline{BC}$. Find the length of $\overline{BX}$.

8. What are the last two digits of the simplified value of $1! + 3! + 5! + \cdots + 2009! + 2011!$?

9. How many terms are in the simplified expansion of $(L + M + T)^{10}$?

10. Ben draws a circle of radius five at the origin, and draws a circle with radius 5 centered at $(15, 0)$. What are all possible slopes for a line tangent to both of the circles?

2 Long Answer Section [130]

Write up full solutions on the provided answer sheets. You are allowed to use the results of earlier problems in the section for later ones, even if you have not solved the earlier problems, but not vice versa.

2.1 Nested Radicals [60]

Note: The two parts of problem 1 are separate problems

1. [**10**] Find integers a and b such that

   (a) $\sqrt{a} + \sqrt{b} = \sqrt{8 + \sqrt{60}}$,

   (b) $\sqrt{a} + \sqrt{b} = \sqrt{14 + \sqrt{192}}$.

   (Hint: Square both sides of the equation.)

2. [**15**] Let a, b, p, q be positive integers such that $a + \sqrt{b} = p + \sqrt{q}$, where q is not a perfect square.

   (a) Explain why $\sqrt{q} - \sqrt{b}$ is equal to some integer n. By solving for b in terms of n and q and using the fact that $\sqrt{q}$ is irrational if q is not a square, show that $n = 0$.

   (b) Conclude that $q = b$ and that $a = p$.

3. [**15**] Consider the equation $\sqrt{a} + \sqrt{b} = \sqrt{p + \sqrt{q}}$, where a, b, p, q are positive integers and q is not a perfect square. Use the previous parts to find p and q in terms of a and b, and conclude that q is divisible by 4.

4. [**20**] By using results from the previous parts and considering the same equation, show that for a and b to be positive integers, it must be the case that $p^2 - q$ is a perfect square. (Hint: Find an expression in terms of a and b for $p^2 - q$.)

2.2 Kinematics [70]

Consider an object that is moving in a straight line. We describe its motion using three quantities.

Quantity 1: Position, which we measure in meters, simply tells us where along the line the object is. We make a choice for the origin, and if our object's position is at 2 meters, we know that it is 2 meters away from the origin in the positive direction. If the position is at -3 meters, then it is 3 meters away from the origin in the negative direction.

Quantity 2: Velocity measures the speed at which the object is moving, in meters per second, and in addition tells us in which direction the object is moving. A positive velocity means that the object is moving forward; a negative velocity means that the object is moving backward. For example, if an object is moving at -5 meters per second, it is moving backwards at a rate such that if the object were to maintain that rate, it would move 5 meters backward in one second.

Quantity 3: Acceleration is probably the least familiar - it measures the *change* in velocity per unit time, and we measure it in meters per second per second, or meters per second squared. If an object has a velocity v_t at time t and v_0 at time 0, where the velocities are measured in meters per second and time is measured in seconds, then the acceleration a is equal to $\frac{v_t - v_0}{t}$. In the following problems, we will assume that this value will always be the same regardless of the value of t; that is, acceleration is constant throughout the motion. Remember that since velocities have signs, these must be plugged in to our expression for a.

In addition to the previous definitions, the following are facts you may use without proof.

- The familiar fact that $d = vt$ applies, with a couple of caveats. We will often deal with situations in which the rate or velocity changes, so this formula will hold true when v is the *average* velocity over the time period. Also, d will be the *displacement*, that is, its final position minus its initial position, of the moving object, rather than the distance it travels. This is an important distinction, because if an object moves forward 5 meters, then backward 5 meters, it has traveled a distance of 10 meters in total, but the object ends up where it started, so its displacement is 0 meters. As usual, t represents time over which we measure the object's motion, measured in seconds.
- We will deal with situations in which acceleration is constant. In this case, our average velocity is simply the average of the starting and ending velocities over our time period. That is, if v_t is our velocity at time t seconds and v_0 is our velocity at time zero seconds (the beginning of our time frame), then the average velocity over this period of t seconds is $\frac{1}{2}(v_t + v_0)$.

5. [**20**] A car accelerates at a constant rate a, measured in meters per second squared. In a time t, measured in seconds, its velocity starts at v_0 and ends at v_t, both measured in meters per second, and the car experiences a displacement d in this time. Show that $d = \frac{1}{2}at^2 + v_0 t$, by using the formulas for acceleration and average velocity.
6. [**20**] Using the same notation of the previous problem, show that $v_t^2 - v_0^2 = 2ad$.
7. [**30**] On the earth, a ball that is thrown straight up or straight down always accelerates with a constant acceleration of -10 meters per second per second, where the negative direction is taken to be downwards, which is true whether the ball is going up or down (note that the ball can be moving upwards, but have a negative acceleration if it is slowing down). A ball is fired upward from the ground at an initial velocity of 20 meters per second, and it eventually comes back down. How long does it take for the ball to hit the ground again, and what is the velocity of the ball at the instant it lands?