2nd Annual Lexington Mathematical Tournament
Individual Round

April 2, 2011

1. A positive integer is randomly selected from among the first 2011 primes. What is the probability that it is even?

2. Julia and Hansol are having a math-off. Currently, Julia has one more than twice as many points as Hansol. If Hansol scores 6 more points in a row, he will tie Julia's score. How many points does Julia have?

3. Segment $\overline{PQ}$ is drawn and squares $ABPQ$ and $CDQP$ are constructed in the plane such that they lie on opposite sides of segment $\overline{PQ}$. If $PQ = 1$, find BD.

4. What is the sum of the 2011 integers closest in value to 0, including 0 itself?

5. The unit cost of a screw is listed as 0.2 cents. When a group of screws is sold to a customer, the total cost of the screws is computed with the listed price and then rounded to the nearest cent. If Al has 50 cents and wishes to only make one purchase, what is the maximum possible number of screws he can buy?

6. Define a sequence by $a_1 = a_2 = 1$, $a_3 = 2$, and

$$a_n + a_{n-3} = a_{n-1} + a_{n-2}$$

for all $n > 3$. What is the value of a_7?

7. A triangle ABC has side lengths $AB = 8$ and $BC = 10$. Given that the altitude to side $\overline{BC}$ has length 4, what is the length of the altitude to side $\overline{AB}$?

8. There are four entrances into Hades. Hermes brings you through one of them and drops you off at the shore of the river Acheron where you wait in a group with five other souls, each of which had already come through one of the entrances as well, to get a ride across. In how many ways could the other five souls have come through the entrances such that exactly two of them came through the same entrance as you did? The order in which the souls came through the entrances does not matter, and the entrance you went through is fixed.

9. Let $ABCD$ be a rhombus and suppose E and F are the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively. If G is the intersection of $\overline{AC}$ and $\overline{EF}$, find the ratio of the area of AEG to the area of $AGFB$.

10. All of the digits of a seven-digit positive integer are either 7 or 8. If this integer is divisible by 9, what is the sum of its digits?

11. Let $ABCD$ be a convex quadrilateral with $AB = AD$, $m\angle A = 40^\circ$, $m\angle C = 130^\circ$, and $m\angle ADC - m\angle ABC = 20^\circ$. Find the measure of the non-reflex angle $\angle CDB$ in degrees.

12. In a round robin tournament of 7 people, each person plays every other person exactly once in a game of table tennis. For each game played, the winner is given 2 points, the loser is given 0 points, and in the event of a tie, each player gets 1 point. At the end of the tournament, what is the average score of the 7 people?

13. Find the second smallest positive integer n such that when n is divided by 5, the remainder is 3, and when n is divided by 7, the remainder is 4.

14. Let L, E, T, M and O be digits that satisfy

$$\begin{array}{cccc} \mathrm{L} & \mathrm{E} & \mathrm{E} & \mathrm{T} \\ + & \mathrm{L} & \mathrm{M} & \mathrm{T} \\ \hline \mathrm{T} & \mathrm{O} & \mathrm{O} & \mathrm{L} \end{array}$$

Given that O has the value of 0, digits may be repeated, and $L \neq 0$, what is the value of the 4-digit integer ELMO?

15. Given that $20N^2$ is a divisor of 11!, what is the greatest possible integer value of N?

16. A *magic square* is a 3×3 grid of numbers in which the sums of the numbers in each row, column, and long diagonal are all equal. How many magic squares exist where each of the integers from 11 to 19 inclusive is used exactly once and two of the numbers are already placed as shown below?

		18
	15	

17. Let ABC be a triangle with $AB = 15$, $AC = 20$, and right angle at A. Let D be the point on $\overline{BC}$ such that $\overline{AD}$ is perpendicular to $\overline{BC}$, and let E be the midpoint of $\overline{AC}$. If F is the point on $\overline{BC}$ such that $\overline{AD} \parallel \overline{EF}$, what is the area of quadrilateral $ADFE$?

18. Let x and y be distinct positive integers below 15. For any two distinct numbers a, b from the set $\{2, x, y\}$, $ab + 1$ is always a positive square. Find all possible values of the square $xy + 1$.

19. A positive six-digit integer begins and ends in 8, and is also the product of three consecutive even numbers. What is the sum of the three even numbers?

20. In the figure below, circle O has two tangents, $\overline{AC}$ and $\overline{BC}$. $\overline{EF}$ is drawn tangent to circle O such that E is on $\overline{AC}$, F is on $\overline{BC}$, and $\overline{EF} \perp \overline{FC}$. Given that the diameter of circle O has length 10 and that $CO = 13$, what is the area of triangle EFC?