1st Annual Lexington Mathematical Tournament
Theme Round

Solutions

1. Answer: $\boxed{42}$
Solution: J can distribute cheetahs equally in both 6 and 7 drawers, so the number of cheetahs that he has is a multiple of both 6 and 7. The least common multiple of 6 and 7 is 42.

2. Answer $\boxed{6}$
Solution: Note that 53/10 is between 5 and 6. This means that it's impossible to distribute the 53 cheetahs among 10 cages such that each cage has at most 5 cheetahs (or else we get at most 50 cheetahs total). Thus, $A > 5$. However, J can put 5 in each cage with 3 left over, and pick 3 cages in which we can add one more cheetah each. This gives J three cages with 6 and seven with 5, so it is possible to get $A = 6$.

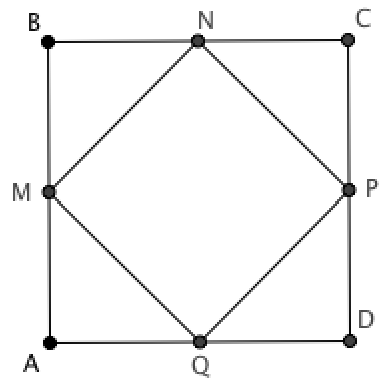
3. Answer $\boxed{41}$
Solution: Say J has m male cheetahs and f female cheetahs. There are 96 cheetahs total, so $m + f = 98$. The second part of the problem gives us $3m = 9 + 2f$. Now, there are many ways we can solve these equations. One way is to multiply the first equation by 3 to get $3m + 3f = 294$, or $3m = 288 - 3f$. Thus, $9 + 2f = 294 - 3f$, from which we get $5f = 285$ and $f = 57$. Using $m + f = 98$, we get $m = 41$.

4. Answer: $\boxed{\text{8:13 PM}}$
Solution: In the first hour, J travels 10 miles, while the cheetah travels 80 miles. Thus, the distance between them is 70 miles. However, once J switches direction, they then run toward each other, and the distance between them is $720 - 70 = 650$ miles on the other side of the loop. The speed at which they are getting closer to each other is $10 + 80 = 90$ miles per hour, since each our, J travels 10 miles and the cheetah travels 80 miles, closing the gap by 90 miles. The time it takes to travel 650 miles at 90 miles per hour is $\frac{650}{90} = 7\frac{2}{9}$ hours, or 7 hours and $60 \cdot \frac{2}{9} = 13\frac{1}{3}$ minutes. Rounded to the nearest minute, this is 7 hours and 13 minutes, and since it takes an hour for J to switch direction, it is 8:13 PM.

5. Answer: $\boxed{115}$
Solution: The number of seats in each row is: 10, 13, 16, 19, 22, 25, 28, 31, 34, 37. We now have to figure out the maximum number of cheetahs that can fit in each row. To get this, we first start with some small cases. If there are two seats, in a circular row, we can fit one cheetah. If there are three, we can fit one, since if we have two, they will have to be sitting next to each other. With a circular row of 4, we can fit two cheetahs, sitting opposite each other. We see that this pattern continues: 1, 1, 2, 2, 3, 3, etc. In general, if the number of seats in a row is even, the number of cheetahs we can fit is half of the number of seats, by placing a cheetah in every other seat. If the number of seats is odd, we can try placing a cheetah in every other seat, but when we come back around to where we started, we get two empty seats, neither of which we can fill, so we subtract 1 from the number of seats and divide by 2. Doing this for our stadium, we get a total of $5 + 6 + 8 + 9 + 11 + 12 + 14 + 15 + 17 + 18 = 115$.

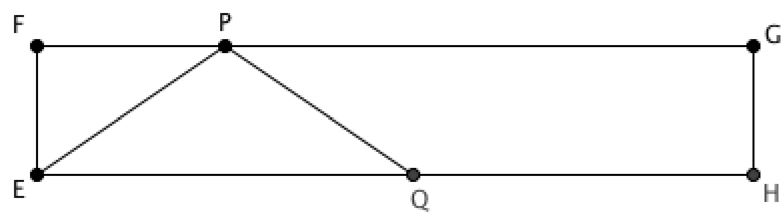
6. Answer: $\boxed{1/2}$

Solution: Let the laser hit CD and DA at P and Q. It looks as if $MNPQ$ is a square, with the four vertices at the midpoints of the edges of square $ABCD$. This would make its area $\frac{1}{2}$, which we can see since the four 'corner' triangles would be congruent with area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, and the area we want would be the area of $ABCD$ minus the sum of the areas of these triangles, which is $1 - 4 \cdot \frac{1}{8} = \frac{1}{2}$.



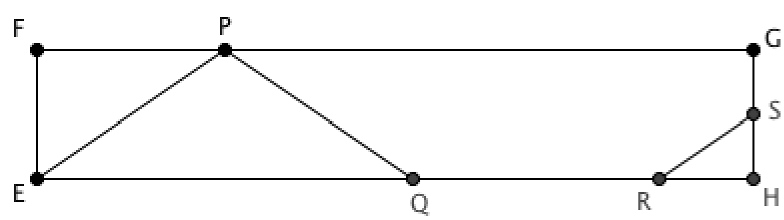
Why are M, N, P, Q midpoints? To start, we're given that M and N are the midpoints of their respective sides. Thus, $AM = MB = BN = NC$. In particular, notice that BMN is a 45-45-90 triangle. Using the properties of lasers, this means that $\angle CNP = \angle BNM = 45°$, and it follows that triangle CNP is 45-45-90 as well. Thus, $NC = CP$, and CP is half of CD, meaning P is the midpoint of CD. Similarly, Q is the midpoint of DA.

7. Answer: $\boxed{50}$



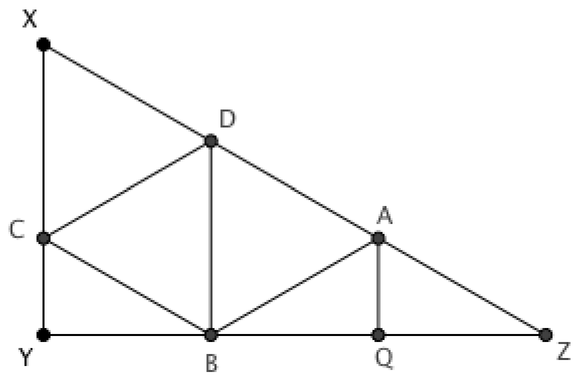
Solution: Applying the Pythagorean Theorem to right triangle EFP, $EP = \sqrt{3^2 + 4^2} = 5$. Now, if we let point where the laser next hits EH be Q, we see that $PQ = EP = 5$. Furthermore, by congruent triangles (where are they? hint: you might have to draw a perpendicular line from Q to FG), the laser has moved four more units to the right. Thus, for each four units it moves to the right, it travels five units in total, and since it has to move a total of 40 units to the right, it travels a total of $\frac{40}{4} \cdot 5 = 50$ units (this is the sum of 10 segments of length 5).

8. Answer: $\boxed{5025/2}$



Solution: This has essentially the same configuration as before, with a twist; the laser will not end at a vertex of the rectangle. This is because it hits one of the horizontal edges of the rectangle, as we saw before, after every 4 units of moving to the right, but 2010 is not divisible by 4. We will get $2008/4 = 502$ segments of length 5, but the laser still has to cover a distance of 2 units to reach edge GH. Still, the laser will bounce off EH (why EH, and not FG?) at some point R with equal angles, and end somewhere on segment GH, say at S. Then, we see that triangle HRS is similar to triangle FPE with ratio $1/2$, since $HR = 2$ and $FP = 4$. Thus, since $EP = 5$, we get $SR = 5/2$, and the total path-length will be $502 + 5/2 = 5025/2$ (note that this is still $5/4$ of 2010).

9. Answer: $\boxed{9}$



Solution: Applying the Pythagorean Theorem, $YZ = \sqrt{2^2 - 1^2} = \sqrt{3}$. It follows that XYZ is a 30-60-90 triangle. Our strategy will be to systematically determine the points of intersection of the laser with the sides of the triangle, relative to the vertices. From Q, the laser will hit XZ at some point A, so we get a right triangle AQZ. This is 30-60-90, because $\angle AZQ = 30°$. This means that $\angle XAB$, where the laser next hits YZ at B, is equal to $\angle ZAQ = 60°$. Now, this gives us two congruent 30-60-90 triangles AQZ and AQB, so $ZQ = QB$, and $QB = BY$, since ZQ is $1/3$ the length of YZ (so all three segments are $1/3$ the length of YZ).

Next, the laser will hit a point C on XY. Again, we get a 30-60-90 triangle CYB (why?), and we see that YC is $1/3$ of YX, since YCB is similar to YXZ with ratio $1/3$ (as they are both 30-60-90). Then, the laser hits XZ at a point D. Now, we have $\angle XCD = \angle BCY = 60°$, and it is given that $\angle DXC = 60°$, so it follows that XDC is an equilateral triangle, and in particular, $\angle XDC = 60°$. Also, $XC = XY - CY = 2/3 = XD$, which is $1/3$ of XZ.

Now, it turns out that the laser's next stop will in fact be point B. This is because $\angle ZDB = \angle XDC = 60°$, so the laser's path from D to YZ will run perpendicular to YZ, as we will get another 30-60-90 triangle with Z, D, and the intersection point of YZ and the laser. Since XD and YB are both $1/3$ of their respective sides, by similar triangles, the laser will in fact hit B.

Now, we see that since $DB \perp YZ$, the laser will bounce back to D, then backwards along the path: C, B, A, then back to Q. The path is thus $QABCDBDCBAQ$, with a total of 9 bounces in between.

10. Answer: $\boxed{1206}$

Solution: We saw from before that the laser's path will be $QABCDBDCBAQ$. Once it gets back to Q, since $AQ \perp YZ$, the laser will bounce back to A, and the cycle will re-start. Adding back the bounce at Q, we get 10 bounces in a cycle, and we want a total of 2010 bounces so we want to find the distance traveled in 201 cycles. We'll instead compute the distance travelled in 1 cycle and multiply by 201. Using similar logic to before and our 30-60-90 triangles, we find that $AD = 1/3$ and $AB = BD = DC = CB = 2/3$. In each cycle, the laser will travel along AQ twice and along the edges of length 2/3 $10-2=8$ times, for a total length of $2(1/3)+8(2/3)=6$. Thus, our answer is $201\cdot 6=1206$.

11. Answer: 1/3

Solution: There are 3 possible outcomes of Bob's play, each with equal probability, and Al wins on one of them, when Bob plays scissors, giving $\frac{1}{3}$.

12. Answer: 1/3

Solution: There are 9 possible combinations of Al and Bob's plays, since each has 3 choices, and furthermore these all have equal probability. There are 3 ways for them to tie, giving a probability of $\frac{3}{9}=\frac{1}{3}$.

13. Answer: 2/729

Solution: From the previous problem, the probability of a tie in a given game is $\frac{1}{3}$, so the probability that someone wins is $\frac{2}{3}$. The first five games must result in ties, and the sixth must have someone winning, so we multiply $(\frac{1}{3})^5(\frac{2}{3})$ to get a probability of $\frac{2}{729}$.

14. Answer: 2/9.

Solution: To do this, we think about how to *construct* a perfect game. We first need to know which pairs of players we'll have; who will play the same thing as whom. There are three ways to decide this, because Al can play the same thing as either Bob, Carl, or D'Angelo, and the second pair will be forced. Now, we have two distinguishable pairs: the one that Al is in, and the one that Al is not in. The first pair has 3 choices for what both player will play, and the other pair has 2 remaining choices, we cannot have all four play the same thing. Thus, there are $3\cdot 3\cdot 2=18$ perfect games. There are $3\cdot 3\cdot 3\cdot 3=81$ possible games, so the probability is $\frac{18}{81}=\frac{2}{9}$.

15. Answer: 3/128

Solution: We first count the number of ways to make the game balanced. This happens if and only if each of Z,Y,X,W,V beats two others. We can represent this by drawing five points Z,Y,X,W,V in a pentagon, and drawing all of its ten sides and diagonals, representing each pair of plays, and drawing an arrow from a point A to a point B if B beats A. Each vertex, then, has two arrows going into it and two arrows going out of it.

Start with the vertex at Z, and label the other four vertices A,B,C,D such that there are arrows going from A to Z, B to Z, Z to C, and Z to D. Now, also make this labeling such that we have an arrow going from A to B and an arrow going from C to D. If either of these does not happen, we can just switch A and B or C and D. We now have four arrows to draw, on the segments AC,BD,BC,BD. Since we have two vertices coming out of A, the other two have to go in to A, so we need arrows from C to A and D to A. Similarly, arrows go into D from Z and C, so we need to draw arrows from D to A (which we've already done) and arrows from D to B. Finally, we need an arrow between B and C, but we see that C has arrows going to A and D, so we need to draw the arrow from B to C. We see that this is also consistent with the arrows at B.

This tells us that given Z,A,B,C,D, and arrows ZA,ZB,CZ,DZ,AB,CD, there is exactly one way to complete the diagram. Now, there are $4!=24$ ways to match A,B,C,D with V,W,X,Y, all of which we see result in different 'pecking orders'. Finally, there are 10 pairs, and 2 ways to assign a direction on each arrow, so $2^{10}=1024$ different pecking orders total. This gives us our probability of $\frac{24}{1024}=\frac{3}{128}$.